A note on online hypercube packing

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Abstract In this paper, we study an online multi-dimensional bin packing problem where all items are hypercubes. Hypercubes of different size arrive one by one, and we are asked to pack each of them without knowledge of the next pieces so that the number of bins used is minimized. Based on the techniques from one dimensional bin packing and specifically the algorithm Super Harmonic by Seiden (J ACM 49:640–671, 2002), we extend the framework for online bin packing problems developed by Seiden to the hypercube packing problem. To the best of our knowledge, this is the first paper to apply a version of Super Harmonic (and not of the Improved Harmonic algorithm) for online square packing, although the Super Harmonic has been already known before. Note that the best previous result was obtained by Epstein and van Stee (Acta Inform 41(9):595–606, 2005b) using an instance of Improved Harmonic. In this paper we show that Super Harmonic is more powerful than Improve Harmonic for online hypercube packing, and then we obtain better upper bounds on asymptotic competitive ratios. More precisely, we get an upper bound of 2.1187 for
square packing and an upper bound of 2.6161 for cube packing, which improve upon the previous upper bounds 2.24437 and 2.9421 (Epstein and van Stee in Acta Inform 41(9):595–606, 2005b) for the two problems, respectively.

**Keywords** Online algorithms · Hypercube packing · Competitive ratio · Approximation algorithms · Harmonic algorithm

1 Introduction

The classical bin packing problem is one of basic problems in computer science and combinatorial optimization etc. There are a lot of applications in real world. In this paper, we study multi-dimensional bin packing problems, which is also related to geometry packing problems and harder than the classical bin packing problem. The Multi-dimensional bin packing problem is defined as below.

**Input:** An input list $L$ of items, where each of items is a $d$-dimensional box, $d \geq 1$ is an integer, and in each dimension, the side length of an item is bounded by one.

**Output:** Pack all the items in $L$ into hypercube bins and minimize the number of bins used such that (i) no two items in a bin overlap with each other, (ii) item’s sides are parallel to bin’s sides. Note that for $d = 1$, the problem degenerates to the classical one-dimensional bin packing problem.

In the following, we focus on a special case of multi-dimensional bin packing problem, i.e., online hypercube packing problem, in which each of items is a hypercube and items arrive one by one, each piece must be immediately and irrevocably assigned to a $d$-dimensional unit bin without any information on subsequence items.

**Asymptotic competitive ratio:** We use the standard asymptotic competitive ratio to evaluate an online algorithm for bin packing problem. Given an input list $L$, let $A(L)$ be the cost (number of bins used) obtained by applying algorithm $A$ to $L$ and let $OPT(L)$ be the corresponding optimal value. Then the asymptotic competitive ratio for algorithm $AR_A^\infty$ is defined as below:

$$R^\infty_A = \lim_{k \to \infty} \sup_{L} \max_{k} \{A(L)/OPT(L) | OPT(L) = k\}.$$

**Previous results:** The classical one-dimensional online bin packing problem has been extensively studied. Johnson et al. (1974) showed that the First Fit algorithm has the competitive ratio 1.7. Yao (1980) improved the upper bound of the competitive ratio to 5/3. Lee and Lee (1985) presented an algorithm with the competitive ratio of 1.63597. Ramanan et al. (1989) improved the upper bound to 1.61217. The best known upper bound is 1.58889 Seiden (2002). On the lower bounds, Yao (1980) showed that no online algorithm has performance ratio less that 1.5. Brown (1979) and Liang (1980) independently improved this lower bound to 1.53635. The current championship of lower bound is 1.54014 Vliet (1992).

On the online hypercube packing problem, Coppersmith and Paghavan (1989) showed an upper bound of $43/16 = 2.6875$ for online square packing and an upper