Local behavior of an iterative framework for generalized equations with nonisolated solutions

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Abstract. An iterative framework for solving generalized equations with nonisolated solutions is presented. For generalized equations with the structure $0 \in F(z) + T(z)$, where $T$ is a multifunction and $F$ is single-valued, the framework covers methods that, at each step, solve subproblems of the type $0 \in A(z, s) + T(z)$. The multifunction $A$ approximates $F$ around $s$. Besides a condition on the quality of this approximation, two other basic assumptions are employed to show $Q$-superlinear or $Q$-quadratic convergence of the iterates to a solution. A key assumption is the upper Lipschitz-continuity of the solution set map of the perturbed generalized equation $0 \in F(z) + T(z) + p$. Moreover, the solvability of the subproblems is required. Conditions that ensure these assumptions are discussed in general and by means of several applications. They include monotone mixed complementarity problems, Karush-Kuhn-Tucker systems arising from nonlinear programs, and nonlinear equations. Particular results deal with error bounds and upper Lipschitz-continuity properties for these problems.

Key words. generalized equation – nonisolated solutions – Newton’s method – superlinear convergence – upper Lipschitz-continuity – mixed complementarity problem – error bounds

1. Introduction

Let a continuous map $F : \mathbb{R}^{\ell_1} \to \mathbb{R}^{\ell_2}$ and a closed multifunction $T : \mathbb{R}^{\ell_1} \rightrightarrows \mathbb{R}^{\ell_2}$ be given and consider the problem of finding $z \in \mathbb{R}^{\ell_1}$ so that

$$0 \in F(z) + T(z). \quad (1)$$

This problem is known as generalized equation. Throughout the paper we assume that its solution set is nonempty, i.e.

$$\Sigma_* := \{z \in \mathbb{R}^{\ell_1} \mid 0 \in F(z) + T(z)\} \neq \emptyset. \quad (2)$$

Generalized equations serve as a general tool for describing, analyzing, and solving different problems in a unified manner. For example, systems of equations and inequalities, several optimality conditions, complementarity problems, or variational inequalities can be reformulated as generalized equations. Newton’s method in its various modifications plays a central role for designing efficient solution algorithms for the problems just mentioned. To cover many of these modifications we will use a simple iterative framework for solving generalized equations. In contrast to various classical approaches, the
local convergence behavior will be analyzed under assumptions that do not imply the isolatedness of a solution. A key assumption is the upper Lipschitz-continuity of the solution set of a perturbed generalized equation. In this way, it is possible both to obtain new results and to reobtain or improve some recent results for particular problems with nonisolated solutions.

In Sect. 2 we will introduce the iterative framework. Sect. 3 describes and discusses the assumptions that will be used in Sect. 4 for proving local superlinear convergence properties. This framework and its analysis extend and improve the approach suggested in [10, Sect. 2]. Both approaches exploit the upper Lipschitz-continuity condition. In comparison to [10] a broader class of (multi)functions $A$ for approximating $F$ enables us to provide a general way of constructing solvable subproblems (Sect. 3.3). Moreover, several applications can now be dealt with (Sect. 5) in a unified manner.

In Sect. 3.1 we give a quite general error bound condition that ensures the upper Lipschitz-continuity. The opposite is shown (Theorem 2) for the case that the multifunction $T$ in the generalized equation is the normal cone to a box. These results are of interest since both upper local Lipschitz-continuity and error bounds play an important role for achieving superlinear convergence in several papers. The earliest reference [18] we are aware of analyzes a proximal point algorithm for maximal monotone generalized equations is analyzed; for recent works see [9, 10, 12, 27–29, 31, 32].

Applications of the framework developed in Sect. 2–4 to particular problems with nonisolated solutions are presented in Sect. 5. We first deal with monotone mixed complementarity problems (MCP). In particular, an Algorithm is suggested that applies certain prox-regularized subproblems. These subproblems are strongly monotone linear MCP. Without further assumptions (except usual smoothness of the problem function) we show that this Algorithm fits our framework and has a local Q-quadratic rate of convergence. Recently, a version of the proximal point algorithm has been introduced [31]. The subproblems used there are linear equations and Q-superlinear convergence is shown if the limit point satisfies the strict complementarity condition.

In Sect. 5.3 we consider Karush-Kuhn-Tucker (KKT) systems arising from nonlinear programs. Firstly, we deal with systems that have nonunique dual solutions due to degenerate active constraints at a solution of the program. For this type of nonisolated solutions several approaches exist, see [10, 12, 17, 19, 28, 29] and [21, 22] for KKT systems arising from variational inequalities. We also show (Sect. 5.3.1) that the so-called stabilized SQP method suggested in [29] and further analyzed in [12, 17, 28] fits our framework. Another class of KKT systems dealt with has nonisolated primal solutions. For this class we derive an algorithm and its Q-quadratic convergence in Sect. 5.3.2.

A third application (Sect. 5.4) is concerned with solving nonlinear equations by a least squares approach. The subproblems we use are similar to those which occur in a Levenberg-Marquardt type method [32]. Despite this similarity and the same convergence results as in [32], the approach in Sect. 5.4 is new. In particular, a relation between an error bound for the nonlinear function and the gradient of its squared norm is given. Moreover, it seems that our subproblems are numerically more stable.

For these particular applications we provide conditions under which the assumptions given in Sect. 3 for achieving Q-quadratic convergence to a solution are satisfied. While doing this, we have obtained several results on error bounds that are new or generalize known ones and that are therefore be of interest on their own. For example, some basic