Abstract. We survey what is known about the information transmitting capacities of quantum channels, and give a proposal for how to calculate some of these capacities using linear programming.

1. Introduction

In this paper, we discuss the capacity of quantum channels. Information theory says that the capacity of a classical channel is essentially unique, and is representable as a single numerical quantity, which gives the amount of information that can be transmitted asymptotically per channel use [46, 15]. Quantum channels, unlike classical channels, do not have a single numerical quantity which can be defined as their capacity for transmitting information. Rather, quantum channels appear to have at least four different natural definitions of capacity, depending on the auxiliary resources allowed, the class of protocols allowed, and whether the information to be transmitted is classical or quantum.

In this paper, we first introduce the background necessary for understanding the capacity of quantum channels, and then define several capacities of these channels. For two of these channel capacities, we sketch possible techniques for computing them which we believe will be more efficient than techniques currently used. These capacities are both reducible to optimization problems over matrices. We believe that a combination of linear programming techniques, including column generation, and non-linear optimization will provide a more efficient method for calculating these capacities. Unfortunately, at the time of writing this paper, I have not yet tested these techniques experimentally. Since I cannot prove that these techniques are efficient, the proof of this pudding must be in the computing, and is thus not yet demonstrated. We hope to test these techniques in the near future.

To date, the means used for numerical computations of quantum channel capacities have been fairly straightforward, often using gradient descent techniques [40]. More research has been done on the calculation of the entanglement of formation [51, 4], a related problem [36]. None of these programs have used combinatorial optimization techniques. For one of the capacities discussed in this paper – the entanglement-assisted capacity – this technique may be fairly efficient, as this capacity has a single local optimum which is also a global optimum. For two other capacities discussed in this paper – the $C_{1,1}$ and $C_{1,\infty}$ capacities – I propose techniques involving linear programming
that could be used for the capacity computation, and which I suspect are much more efficient than straightforward optimization. For another capacity – the one-way quantum capacity – there are multiple local maxima in the optimization problem, and we need to determine the global maximum. In this case, unfortunately, although hill climbing does not seem like it would be efficient, I do not have any alternative techniques to suggest.

This paper originates in my research investigating the capacities of a quantum channel [48]. In order to show that a certain channel capacity (which I do not deal with in this paper; it is less natural than the capacities covered here) lies strictly between two other channel capacities, I needed to calculate some of these capacities. Specifically, I needed to calculate what I call the $C_{1,1}$ capacity of a fairly simple quantum channel. I realized that this was a problem which could be solved numerically using linear programming, and I used this technique to obtain a picture of the $C_{1,1}$ capacity landscape which was satisfactory for my application. During this computation, it became clear that a better way to solve this problem would be to use column generation techniques to make the linear program more efficient, and that these would furthermore also be useful for computing other capacities of quantum channels. I have not yet had time to experimentally test these new techniques (rather, my program started with enough columns to ensure obtaining a close approximation of the capacity; this would be an enormous waste of resources for larger problems, but for my purposes it was quite adequate). This paper will explain the column generation technique. I will try to make it comprehensible both to researchers with background in mathematical programming and to researchers with background in quantum information theory. Those wishing more background information on linear programming, on quantum computing and information, or on classical information theory can find them in textbooks such as [14, 38, 15]. More specifically, I will give proposals for how to compute two capacities for carrying classical information over a quantum channel: namely, the $C_{1,1}$ capacity and the $C_{1,\infty}$ capacity. These techniques should also work for computing a formula that I conjecture gives the classical entanglement-assisted capacity with limited entanglement; this extrapolates between the $C_{1,\infty}$ capacity and the entanglement-assisted capacity. The description of quantum information theory and capacities contained in here is largely taken from the paper [47].

2. Quantum information theory

The discipline of information theory was founded by Claude Shannon in a truly remarkable paper [46] which laid down the foundations of the subject. We begin with a quote from this paper which a nutshell summarizes one of the main concerns of information theory:

*The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point.*

This paper proposed the definition of the capacity of a classical channel as the amount of information per channel use that can be transmitted asymptotically in the limit of many channel uses, with near perfect reproduction at the receiver’s end, and gave a simple and elegant formula for the capacity. Here, the information is the logarithm (base 2) of the