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A simplicial branch-and-bound algorithm for solving quadratically constrained quadratic programs

Received: May 20, 2004 / Accepted: February 3, 2005
Published online: April 28, 2005 – © Springer-Verlag 2005

Abstract. We propose a branch-and-bound algorithm for solving nonconvex quadratically-constrained quadratic programs. The algorithm is novel in that branching is done by partitioning the feasible region into the Cartesian product of two-dimensional triangles and rectangles. Explicit formulae for the convex and concave envelopes of bilinear functions over triangles and rectangles are derived and shown to be second-order cone representable. The usefulness of these new relaxations is demonstrated both theoretically and computationally.

Key words. Nonconvex quadratic programming – Global optimization – Convex envelope – Branch-and-bound

1. Introduction

In this paper, we discuss branch-and-bound methods for solving the quadratically constrained quadratic program (QCQP) which can be written as

$$\min_{x \in \mathbb{R}^n} \{ q_0(x) \mid q_k(x) \geq b_k \forall k \in M, l_i \leq x_i \leq u_i \forall i \in I \},$$

where $q_k = c_k^T x + x^T Q_k x \forall k \in M \cup \{0\}$, and $I = \{1, 2, \ldots, n\}$. We assume that explicit lower bounds and upper bounds on $x_i$ are known, so that the feasible region is compact. We do not assume any convexity properties of the $q_k(x)$, so the objective function of the problem may be convex, concave, or indefinite, and the set of feasible solutions need not be convex or connected.

QCQP generalizes many well-known, difficult optimization problems. Linear mixed 0-1 programming, fractional programming, bilinear programming, polynomial programming, and bilevel programming problems can all be written as instances of QCQP, so QCQP is $\mathcal{NP}$-Hard. From a practical standpoint, QCQP is one of the most challenging optimization problems—the current size of instances that can be solved to provable optimality remains very small in comparison to other $\mathcal{NP}$-Hard problem classes such as mixed integer programming.

In this work, we describe a branch-and-bound algorithm for solving QCQP that is based on subdividing the feasible region into the Cartesian product of triangles and rectangles. It can be viewed as an extension of the work of Al-Khayyal and Falk [2], who derive a formula for the convex envelope of a product of variables over a rectangle and give a branch-and-bound algorithm based on the formula. Al-Khayyal [1] extends the
formula for the concave envelope of the product of variables, and Al-Khayyal, Larsen, and Van Voorhis develop a branch-and-bound algorithm based on these relaxations [3]. Raber [24, 25] also gives a simplicial-subdivision based algorithm for QCQP. In Raber’s work, the feasible region is enclosed in a high-dimensional simplex, and this simplex is subdivided in the spirit suggested by Horst [15]. Our work is different in that the feasible region is enclosed in an initial hyper-rectangle, and that hyper-rectangle is subdivided into the Cartesian product of rectangles and triangles (low-dimensional simplices). Sherali and Alameddine [30] use the Reformulation-Linearization Technique (RLT) to solve bilinear programming problems, and Audet et al. [5] extend the use of RLT in solving QCQP by including different classes of linearizations. Kim and Kojima [19] extend the lift-and-project idea of RLT to create a second-order cone programming relaxation for QCQP. DC (Difference of Convex) programming techniques were used by Phong, Tao and Hoai An to solve QCQP, and DC programming techniques form the basis of the general global optimization software $\alpha$BB [4]. BARON [28, 33] is a mature, sophisticated software package that can solve QCQP using convex/concave envelopes (in the spirit of [2]), but augmented with features such as sophisticated range reduction and branching techniques [27, 34]. BARON also can be accessed through a link to the commercial modeling language GAMS.

The paper is organized as follows. In Sect. 2, expressions for the convex and concave envelope of the bilinear function $f(x,y) = xy$ over rectangular and triangular regions are derived. In Section 3 a simple triangle-based branching scheme is introduced, and based upon such a scheme a nonlinear programming relaxation for QCQP is given. Section 4 demonstrates that the nonlinear constraints in the relaxation are representable as second-order cone constraints. A polyhedral outerapproximation to the second-order cone is given, resulting in a new linear programming relaxation to QCQP. Section 5 introduces two measures of the tightness of approximations to the convex and concave envelopes and computes these measures for the envelope expressions derived in Sect. 2. In Sect. 6, computational results are given to show the usefulness of the triangle-based branch-and-bound method.

2. Relaxations for QCQP

Tractable relaxations of the nonconvex problem QCQP can be obtained using the notions of convex envelopes and concave envelopes. For a function $f : \Omega \rightarrow \mathbb{R}$, the convex envelope of $f$ over $\Omega$, denoted $\operatorname{vex}_\Omega(f)$, is the pointwise supremum of convex underestimators of $f$ over $\Omega$. Likewise, the concave envelope of $f$ over $\Omega$, denoted $\operatorname{cav}_\Omega(f)$ is the pointwise infimum of concave overestimators of $f$ over $\Omega$. This paper refers extensively to the convex and concave envelope expressions for the bilinear function $f(x,y) = xy$. To simplify the notation, the following definitions are made:

$$
\begin{align*}
\operatorname{vex}_{xy}\Omega & \equiv \operatorname{vex}_\Omega(xy) \\
\operatorname{cav}_{xy}\Omega & \equiv \operatorname{cav}_\Omega(xy).
\end{align*}
$$