Strong KKT conditions and weak sharp solutions in convex-composite optimization

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Abstract Using variational analysis techniques, we study convex-composite optimization problems. In connection with such a problem, we introduce several new notions as variances of the classical KKT conditions. These notions are shown to be closely related to the notions of sharp or weak sharp solutions. As applications, we extend some results on metric regularity of inequalities from the convex case to the convex-composite case.

Keywords Convex-composite optimization · Strong KKT condition · Sharp solution · Weak sharp solution

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1 Introduction

Let $X$ be a Banach space and $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Recall that $\phi$ is said to have a sharp minimum at $\bar{x} \in X$ if there exist two

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positive numbers $\eta$ and $\delta$ such that

$$\eta \|x - \bar{x}\| \leq \phi(x) - \phi(\bar{x}) \quad \forall x \in B(\bar{x}, \delta),$$

where $B(\bar{x}, \delta)$ denotes the open ball with center $\bar{x}$ and radius $\delta$. Let $\Omega$ be a subset of $X$, $\lambda := \inf \{ \phi(x) : x \in \Omega \}$ and $S := \{ x \in \Omega : \phi(x) = \lambda \}$. Suppose that $\lambda$ is finite. Following Burke and Ferris [5], we say

(a) $S$ is a set of weak sharp minima for $\phi$ over $\Omega$ if there exists $\eta > 0$ such that

$$\eta d(x, S) \leq \phi(x) - \lambda \quad \forall x \in \Omega;$$

(b) $\bar{x} \in \Omega$ is a local weak sharp minimum for $\phi$ over $\Omega$ if there exist $\eta, \delta \in (0, +\infty)$ such that

$$\eta d(x, S(\bar{x})) \leq \phi(x) - \phi(\bar{x}) \quad \forall x \in \Omega \cap B(\bar{x}, \delta),$$

where $S(\bar{x}) := \{ x \in \Omega : \phi(x) = \phi(\bar{x}) \}$. The notions of sharp minima and weak sharp minima have many important consequences for convergence analysis and stability analysis of many algorithms. The readers can look at [3–5,9,25,28,29] and references therein for the history and motivation for the study of sharp minima and weak sharp minima. In terms of normal cones and subdifferentials, Burke and Ferris [5] established some valuable duality characterizations for weak sharp minima in finite dimensional spaces; Burke and Deng [3], with the help of the Fenchel dual technique, extended these results to a infinite dimensional space setting and established results on local weak sharp minima in a Hilbert space; the authors [28], using the Banach space geometrical technique, provided some characterizations for a local weak sharp minimum in a general Banach space. All the works mentioned above are under the convexity assumption. In this paper, we will relax the convexity assumption by considering the “convex-composite” situation, that is, the functions $\phi$ involved are given in the form $\phi = \psi \circ f$, where $f$ is a smooth function from a Banach space $X$ to another Banach space $Y$ and $\psi$ is a convex real-valued function on $Y$. Such a function (which is usually referred to as a convex-composite function) is not necessarily convex but shares many interesting and useful properties with convex functions. The class of such functions is huge (in particular it contains amenable functions due to Poliquin and Rockafellar (see [23, P. 442])) and these functions arise naturally in mathematical programming (see Rockafellar [21] where he gave many interesting examples showing that a wide spectrum of problems can be cast in terms of convex-composite functions).

Let $\phi_0, \ldots, \phi_m$ be proper lower semicontinuous functions on $X$ and let $C$ be a closed set in $X$. Consider the following constrained optimization problem

$$\begin{align*}
\min \phi_0(x) \\
\text{s.t. } & \phi_i(x) \leq 0, \quad i = 1, \ldots, m, \quad (1.1) \\
x \in C.
\end{align*}$$