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A new look at smoothing Newton methods for nonlinear complementarity problems and box constrained variational inequalities

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Abstract. In this paper we take a new look at smoothing Newton methods for solving the nonlinear complementarity problem (NCP) and the box constrained variational inequalities (BVI). Instead of using an infinite sequence of smoothing approximation functions, we use a single smoothing approximation function and Robinson’s normal equation to reformulate NCP and BVI as an equivalent nonsmooth equation

\[ H(u; x/D_0, \ldots) \]

where \( H \) is a parameter variable and \( x \) is the original variable. The central idea of our smoothing Newton methods is that we construct a sequence \( z_k \) such that the mapping \( H(z) \) is continuously differentiable at each \( z_k \) and may be non-differentiable at the limiting point of \( z_k \). We prove that three most often used Gabriel-Moré smoothing functions can generate strongly semismooth functions, which play a fundamental role in establishing superlinear and quadratic convergence of our new smoothing Newton methods. We do not require any function value of \( F \) or its derivative value outside the feasible region while at each step we only solve a linear system of equations and if we choose a certain smoothing function only a reduced form needs to be solved. Preliminary numerical results show that the proposed methods for particularly chosen smoothing functions are very promising.

Key words. variational inequalities – nonsmooth equations – smoothing approximation – smoothing Newton method – convergence

1. Introduction

Consider the variational inequalities (VI for abbreviation): Find \( y^* \in X \) such that

\[ (y - y^*)^T F(y^*) \geq 0 \quad \text{for all } y \in X, \]

where \( X \) is a nonempty closed subset of \( \mathbb{R}^n \) and \( F : D \to \mathbb{R}^n \) is continuously differentiable on some open set \( D \), which contains \( X \). In this paper, unless otherwise stated, we assume that

\[ X := \{ y \in \mathbb{R}^n | a \leq y \leq b \}, \]

where \( a \in [\mathbb{R} \cup \{-\infty\}]^n \), \( b \in [\mathbb{R} \cup \{\infty\}]^n \) and \( a < b \). Then (1) becomes the box constrained variational inequalities (BVI for abbreviation). This assumption is not restrictive because if in (1) the set \( X \) is not of the form (2) but is represented by several

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equalities and inequalities, then under standard constraint qualifications [25] we can equivalently transform (1) into a new VI with the constraint set of form (2), possibly with increased dimension (see, e.g., [54]). When \( X = \mathbb{R}_+^n \), VI reduces to the nonlinear complementarity problem (NCP for abbreviation): Find \( y^* \in \mathbb{R}_+^n \) such that

\[
F(y^*) \in \mathbb{R}_+^n \quad \text{and} \quad (F(y^*))^T y^* = 0. \tag{3}
\]

It is well known (see, e.g., [25]) that solving (1) is equivalent to finding a root of the following equation:

\[
W(y) := y - \Pi_X(y - F(y)) = 0, \tag{4}
\]

where for any \( x \in \mathbb{R}^n \), \( \Pi_X(x) \) is the Euclidean projection of \( x \) onto \( X \) and \( X \) is a nonempty closed convex subset of \( \mathbb{R}^n \), which is not necessarily of the form (2). It is also well known that if \( X \) is a closed convex subset of \( \mathbb{R}^n \), then solving VI is equivalent to solving the following Robinson’s normal equation

\[
E(x) := F(\Pi_X(x)) + x - \Pi_X(x) = 0 \tag{5}
\]

in the sense that if \( x^* \in \mathbb{R}^n \) is a solution of (5) then \( y^* := \Pi_X(x^*) \) is a solution of (1), and conversely if \( y^* \) is a solution of (1) then \( x^* := y^* - F(y^*) \) is a solution of (5) [49]. Both (4) and (5) are nonsmooth equations and have led to various generalized Newton’s methods. See [25], [40], [22] and [18] for a review of these methods.

By using the Gabriel-Moré smoothing function for \( 5\Pi_X \), we can construct approximations for \( E(\cdot) \):

\[
G(u, x) := F(p(u, x)) + x - p(u, x), \quad (u, x) \in \mathbb{R}^m \times \mathbb{R}^n. \tag{6}
\]

where for each \( i \in N := \{1, 2, ..., n\} \), \( p_i(u, x) = q(u_i, a_i, b_i, x_i) \) and for any \((\mu, c, d, w) \in \mathbb{R} \times \mathbb{R} \cup (-\infty) \times \mathbb{R} \cup [\infty) \times \mathbb{R} \) with \( c \leq d \), \( q(\mu, c, d, w) \) is defined by

\[
q(\mu, c, d, w) = \begin{cases} 
\phi([\mu], c, d) \text{ if } \mu \neq 0 \\
\Pi_{[c,d] \cap \mathbb{R}}(w) \text{ if } \mu = 0
\end{cases} \tag{7}
\]

and \( \phi(\mu, c, d, w) \), \( (\mu, w) \in \mathbb{R}_{++} \times \mathbb{R} \) is a Gabriel-Moré smoothing approximation function [23], also, see Sect. 2 for the definition of \( \phi(\cdot) \). For example, for NCP we can take the Chen-Harker-Kanzow-Smale smoothing NCP function \([4, 31, 51]\]

\[
\phi(\mu, 0, \infty, w) = \frac{\sqrt{w^2 + 4\mu^2} + w}{2}, \quad (\mu, w) \in \mathbb{R}_{++} \times \mathbb{R},
\]

which is a special Gabriel-Moré smoothing function. In this paper, unless otherwise stated, we always assume that \( c \in \mathbb{R} \cup \{-\infty\}, d \in \mathbb{R} \cup \{\infty\} \) and \( c \leq d \). By Lemma 2.2