Symplectic Convexity for Orbifolds

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Abstract  Let a connected compact Lie group $G$ act on a connected symplectic orbifold of orbifold fundamental group $\Gamma$. If the action preserves the symplectic structure and there is a $G$-equivariant and mod-$\Gamma$ proper momentum map for the lifted action on the universal branch covering orbifold, and if the lifted $G$-action commutes with that of $\Gamma$, then the symplectic convexity theorem is still true for this kind of lifted Hamiltonian action.

Keywords  momentum map, mod-$\Gamma$ proper map, symplectic orbifold

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1 Introduction

An orbifold is a Hausdorff topological space locally modelled on $\mathbb{R}^n$ modulo finite group actions. If the group actions are trivial, we recover the concept of manifold. Quite an interesting thing is that the enlarged category is closed under the quotients by finite groups. In symplectic geometry, an important construction of symplectic quotients called Marsden–Weinstein quotients, generically, are not manifolds but symplectic orbifolds [1]. Naturally, we would like to generalize some basic results on symplectic manifolds to the orbifold cases.

Atiyah, independently, Guillemin and Sternberg established symplectic convexity theorems for Hamiltonian torus actions on symplectic manifolds in [2–4]. Lerman and Tolman got the orbifold versions in [1]. In this note we give some generalizations of their theorems using different methods.

Theorem 1.1  Let $T$ be a torus and $(M, \omega)$ a connected symplectic $T$-orbifold. Let $\widetilde{M} \rightarrow M$ be the universal branch covering orbifold and $\Gamma = \pi_1^{orb}(M)$ the orbifold fundamental group. Assume there is a momentum map $\tilde{J} : \widetilde{M} \rightarrow \mathfrak{t}^*$ for the lifted action. If $\tilde{J}$ is mod-$\Gamma$ proper and the lifted $T$-action commutes with that of $\Gamma$, then $\tilde{J}(\widetilde{M})$ is a closed convex set and $\tilde{J} : \widetilde{M} \rightarrow \tilde{J}(\widetilde{M})$ is an open, fibre-connected map.

to the setting of Poisson actions of compact Poisson–Lie groups on symplectic manifolds. If it is symplectic action on the orbifold, we have the following extension:

**Theorem 1.2** Let $G$ be a connected compact Lie group and $(M, \omega)$ a connected symplectic $G$-orbifold. Let $\tilde{M} \rightarrow M$ be the universal branch-covering orbifold and $\Gamma = \pi^\text{orb}_1(M)$ the orbifold fundamental group. Assume there is a $G$-equivariant momentum map $\tilde{J} : \tilde{M} \rightarrow g^\ast$. If $\tilde{J}$ is mod-$\Gamma$ proper and the lifted $G$-action commutes with that of $\Gamma$, then $\tilde{J}(M) \cap t_\ast^+$ is a closed convex set and $\tilde{J} \rightarrow \tilde{J}(\tilde{M})$ is a fibre-connected map.

There are several methods to prove symplectic convexity theorems. Atiyah, Guillemin and sternberg, Lerman and Tolman, Kirwan in [1–4, 6] employed Morse theory. It is easy to show, using a normal form for Hamiltonian action, the momentum map is locally convex [2–3]. The Morse theory gives rise to a global convexity theorem. Hilgert–Neeb–Plank [10] offered another proof by using a ‘local-global-principle’, substituting the assumption that the acted manifold is compact by an assumption that the momentum map is proper. Lerman–Meinrenken–Tolman–Woodward [11] used the symplectic cutting technique. Intuitively, by cutting out infinity and collapsing the incision to a point we get a compact symplectic space such that the original non-compact symplectic space is equivariantly embedded in it as an open submanifold. Thus the proof is reduced to the compact case. This method still works in orbifold cases, and as a result the symplectic convexity theorems are extended to non-compact orbifold cases [11].

In this paper, we use the techniques developed in [10] by Hilgert–Neeb–Plank where the authors dealt with the manifold cases. Their proof is more analytical and elementary than those based on Morse theory. It uses less knowledge of symplectic geometry. In fact, it suffices to know that the momentum map is locally convex, locally open, locally fiber-connected. But these are easily understood if we know the symplectic version of the slice theorem for smooth group actions. Furthermore, this proof tells us clearly what causes the convexity and why it should be so. To some extent, it builds the symplectic convexity theorems on set-theoretic topology.

Here is a brief description of the structure of this paper. In Section 2 we review some basic concepts and explain the connections between symplectic action and Hamilton action. Following the same idea of Hilgert–Neeb–Plank, we define a map quotient for mod-$\Gamma$ proper map in Section 3 and prove that it is a Hausdorff space. Finally we give proofs of Theorem 1.1 and Theorem 1.2 in the last two sections.

## 2 Symplectic Actions and Hamilton Actions

We refer the reader to Chapter 13 of [15] for a nice account of orbifolds and to [1] for definitions of symplectic orbifolds and Hamiltonian actions on them. Let $G$ be a connected Lie group with Lie algebra $g$ and $M$ a smooth connected $G$-orbifold with symplectic structure $\omega$. A smooth action $G \times M \rightarrow M$ is called symplectic if $\omega$ is invariant under the action of $G$. In this case $M$ is called a symplectic $G$-orbifold. A symplectic action is called Hamiltonian action if there exists a map $J : M \rightarrow g^\ast$, called a momentum map, such that

$$i(\xi_M)\omega = dJ_\xi,$$