Averaging Technique for the Oscillation of Second Order Damped Elliptic Equations

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Abstract By using the averaging technique, we establish some oscillation theorems for the second order damped elliptic differential equation

\[ \sum_{i,j=1}^{N} D_i[a_{ij}(x)D_jy] + \sum_{i=1}^{N} b_i(x)D_iy + c(x)f(y) = 0, \]

which extend and improve some known results in the literature.

Keywords oscillation, averaging technique, second order, elliptic differential equation, damped

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1 Introduction

Consider the second order damped elliptic differential equation

\[ \sum_{i,j=1}^{N} D_i[a_{ij}(x)D_jy] + \sum_{i=1}^{N} b_i(x)D_iy + c(x)f(y) = 0 \quad (1.1) \]

in \( \Omega(1) \), where \( N \geq 2 \), \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \), \( |x| = [\sum_{i=1}^{N} x_i^2]^{1/2} \), \( D_i = \partial/\partial x_i \) for all \( i \), and \( \Omega(1) = \{ x \in \mathbb{R}^N : |x| \geq 1 \} \).

Throughout this paper we shall assume that:

(A1) \( A = (a_{ij})_{N \times N} \) is a real symmetric and positive definite matrix function with \( a_{ij} \in C^{1+\nu}_{loc}(\Omega(1), \mathbb{R}) \) for all \( i, j, \nu \in (0, 1) \);

(A2) \( b_i \in C^{4+\nu}_{loc}(\Omega(1), \mathbb{R}) \) for all \( i \), and \( c \in C_0^{4+\nu}(\Omega(1), \mathbb{R}) \);

(A3) \( f \in C(\mathbb{R}, \mathbb{R}) \cup C^1(\mathbb{R} - \{0\}, \mathbb{R}) \), \( f'(y) \geq k > 0 \), \( yf(y) > 0 \) for \( y \neq 0 \).

When \( b_i(x) \equiv 0 \) for all \( i \), we consider

\[ \sum_{i,j=1}^{N} D_i[a_{ij}(x)D_jy] + c(x)f(y) = 0. \quad (1.2) \]

By a solution (classical solution) of (1.1) we mean a function \( y \in C^{2+\nu}_{loc}(\Omega(1), \mathbb{R}) \) which satisfies (1.1) on \( \Omega(1) \). Regarding the question of the existence of solutions of (1.1), we refer the reader to the monograph [1]. We restrict our attention only to the nontrivial solution \( y(x) \) of (1.1), i.e., for any \( a \geq 1 \), \( \sup\{|y(x)| : |x| \geq a\} > 0 \). The oscillation is considered in the usual sense, i.e., a nontrivial solution \( y(x) \) of (1.1) is said to be oscillatory if the set
\{x \in \Omega(1) : y(x) = 0\} is unbounded; otherwise it is said to be nonoscillatory. (1.1) is called oscillatory if all its solutions are oscillatory.

For the corresponding 1-dimensional ordinary differential equation of (1.2),
\[ y''(t) + c(t)y(t) = 0, \]
the most important simple oscillation criteria is the well-known Fite–Wintner theorem [2–3], which states that, if \( c \in C([1, \infty), \mathbb{R}) \) and satisfies
\[ \lim_{t \to \infty} c(t) = \lim_{t \to \infty} \int_1^t c(s)ds = \infty, \]
then (1.3) is oscillatory. In fact, Fite [2] in addition assumed that \( c(t) \) is nonnegative, while Wintner [3] proved a stronger result which required a weaker condition involving the integral average of \( C(t) \), i.e.,
\[ \lim_{t \to \infty} \frac{1}{t} \int_1^t C(s)ds = \infty. \]

The method of the proof of Wintner [3] differs from that of Fite [2] and is the first result based on the integral averages. This technique was explored by Kamenev [4], who proved that, if there exists a number \( \alpha > 1 \) such that
\[ \limsup_{t \to \infty} \frac{1}{t^\alpha} \int_1^t (t-s)^\alpha c(s)ds = \infty, \]
then (1.3) is oscillatory. Kamenev’s criterion was further extended to the damped linear differential equation
\[ (a(t)y'(t))' + b(t)y'(t) + c(t)y(t) = 0, \]
by Yan [5]. Yan [5] also gave another result when a similar condition to that of (1.6) fails. Chen [6] extended Yan’s second result to the damped nonlinear differential equation
\[ (a(t)y'(t))' + b(t)y'(t) + c(t)f(y) = 0. \]
Yan’s theorem has also been extended by Philos [7], for (1.3) using general means. Recently, Wong [8] further improved the results of Yan [5], Chen [6] and Philos [7]. However, all the aforementioned results involve the integral of \( c(t) \) on the whole interval \([1, \infty)\), and hence require the information of \( c(t) \) on the entire half-line \([1, \infty)\). But from the Sturm Separation Theorem, it is clear that the oscillation of (1.3) is only an interval property, i.e., if there exists a sequence of subintervals \([t_i, T_i]\) of \([1, \infty)\), where \( t_i \to \infty \) such that for each \( i \) there exists a solution of (1.3) that has at least two zeros in \([t_i, T_i]\), then every solution of (1.3) is oscillatory, no matter how “bad” (1.3) is (or \( c \) is) on the remaining parts of \([t_0, \infty)\). On the basis of this observation, in 1999, Kong [9] employed the technique of Philos [7] and presented several interval oscillation criteria for (1.3), which involve Kamenev’s type of condition. Kong’s criteria [9] were recently extended to the damped equation (1.8) by Li and Agarwal [10].

For (1.2), there exists a well-elaborated oscillation theory [see, for example, 11–17]. In particular, in 1980, Noussair and Swanson [11] first established the Fite–Wintner-type theorem for (1.2) based on the partial Riccati transformation
\[ w(x) = -\frac{\alpha(|x|)}{f(y(x))} (A\nabla y)(x), \]
where \( \alpha \in C^2(\mathbb{R}^+, \mathbb{R}^+) \) is an arbitrary given function, \( \nabla y \) denotes the gradient of \( y \). The survey paper by Swanson [12] contains a complete bibliography up to 1979. Very recently, using the