Commuting Dual Toeplitz Operators
on the Orthogonal Complement of the Dirichlet Space

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Abstract In this paper we characterize commuting dual Toeplitz operators with harmonic symbols on the orthogonal complement of the Dirichlet space in the Sobolev space. We also obtain the sufficient and necessary conditions for the product of two dual Toeplitz operators with harmonic symbols to be a finite rank perturbation of a dual Toeplitz operator.

Keywords Sobolev space, Dirichlet space, dual Toeplitz operator

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1 Introduction

Let $D$ be the open unit disk in the complex plane $\mathbb{C}$ and let $dA$ denote the Lebesgue measure on $D$, normalized so that the measure of $D$ equals 1. The Sobolev space $W^{1,2}$ consists of functions $u : D \to \mathbb{C}$ with the weak partial derivatives of order 1, for which the norm is

$$\|u\|_{W^{1,2}} = \left( \int_D |u|^2 + \int_D \left| \frac{\partial u}{\partial z} \right|^2 + \left| \frac{\partial u}{\partial \bar{z}} \right|^2 dA \right)^{\frac{1}{2}} < \infty.$$  \hspace{1cm} (1)

The space $W^{1,2}$ is a Hilbert space with the inner product

$$\langle u, v \rangle_{W^{1,2}} = \int_D u dA \int_D \bar{v} dA + \int_D \left\langle \frac{\partial u}{\partial z}, \frac{\partial v}{\partial z} \right\rangle_2 + \left\langle \frac{\partial u}{\partial \bar{z}}, \frac{\partial v}{\partial \bar{z}} \right\rangle_2,$$

where the symbol “$\langle \cdot, \cdot \rangle_2$” denotes the inner product in the Hilbert space $L^2(D, dA)$. The Dirichlet space $D_0$ is the closed subspace of $W^{1,2}$ consisting of all holomorphic functions $f \in W^{1,2}$ with $f(0) = 0$. Then the Dirichlet space $D_0$ is a Hilbert space with inner product

$$\langle u, v \rangle_{D_0} = \left\langle \frac{\partial u}{\partial z}, \frac{\partial v}{\partial z} \right\rangle_2.$$

Each point evaluation in $D$ is easily verified to be a bounded linear functional on $D_0$. Hence, for each $w \in D$, there exists a unique function $K_w \in D_0$ which has the following reproducing property

$$f(w) = \langle f, K_w \rangle_{D_0}$$

for every $f \in D_0$. It is known that the reproducing kernel function $K_w$ is given by

$$K_w(z) = -\log(1 - z \bar{w}) = \sum_{k=1}^{\infty} \frac{z^k \bar{w}^k}{k}.$$
Let $P$ denote the orthogonal projection from $W^{1,2}$ onto $\mathcal{D}_0$. It is well known that $P$ can be represented by the integral formula

$$P(u)(w) = \langle u, K_w \rangle = \int_D \frac{\partial u}{\partial \bar{z}} \frac{\partial K_w(z)}{\partial z} dA(z), \quad u \in W^{1,2}.$$ 

Decomposing $W^{1,2}$ into $\mathcal{D}_0 \oplus \mathcal{D}_0^\perp$, let $Q$ denote the orthogonal projection from $W^{1,2}$ onto $\mathcal{D}_0^\perp$.

Define

$$W^{1,\infty}(D) = \left\{ \varphi \in W^{1,2} : \varphi, \frac{\partial \varphi}{\partial \bar{z}}, \frac{\partial \varphi}{\partial z} \in L^\infty(D) \right\},$$

where $L^\infty(D)$ denotes the algebra of all essentially bounded measurable functions on $D$. For $\varphi \in W^{1,\infty}(D)$, the norm of $\varphi$ is defined by

$$\|\varphi\|_{1,\infty} = \max \left\{ \|\varphi\|_\infty, \left\| \frac{\partial \varphi}{\partial \bar{z}} \right\|_\infty, \left\| \frac{\partial \varphi}{\partial z} \right\|_\infty \right\}. \quad (2)$$

Given a function $\varphi \in W^{1,\infty}(D)$, the Toeplitz operator $T_\varphi(\mathcal{D}_0 \to \mathcal{D}_0)$, the Hankel operator $H_\varphi(\mathcal{D}_0 \to \mathcal{D}_0^\perp)$, the dual Toeplitz operator $S_\varphi(\mathcal{D}_0^\perp \to \mathcal{D}_0^\perp)$ and the operator $R_\varphi(\mathcal{D}_0^\perp \to \mathcal{D}_0)$ with symbol $\varphi$ are defined respectively by

$$T_\varphi(u) = P(\varphi u), \quad u \in \mathcal{D}_0, \quad (3)$$

$$H_\varphi(u) = Q(\varphi u), \quad u \in \mathcal{D}_0, \quad (4)$$

$$S_\varphi(u) = Q(\varphi u), \quad u \in \mathcal{D}_0^\perp, \quad (5)$$

$$R_\varphi(u) = P(\varphi u), \quad u \in \mathcal{D}_0^\perp. \quad (6)$$

For a function $\varphi \in W^{1,\infty}(D)$, we define the multiplication operator $M_\varphi$ to be the operator on $W^{1,2}$ given by $M_\varphi(u) = \varphi u$, for $u \in W^{1,2}$. We will show that $M_\varphi$ is bounded in Lemma 1.

Under the decomposition $W^{1,2} = \mathcal{D}_0 \oplus \mathcal{D}_0^\perp$, the multiplication operator $M_\varphi$ is represented as

$$\begin{pmatrix} T_\varphi & R_\varphi \\ H_\varphi & S_\varphi \end{pmatrix}.$$

The identity $M_\varphi \psi = M_\varphi M_\psi$ implies

$$T_{\varphi \psi} = T_\varphi T_\psi + R_\varphi H_\psi, \quad (7)$$

$$R_{\varphi \psi} = T_\varphi R_\psi + R_\varphi S_\psi, \quad (8)$$

$$H_{\varphi \psi} = H_\varphi T_\psi + S_\varphi H_\psi, \quad (9)$$

$$S_{\varphi \psi} = H_\varphi R_\psi + S_\varphi S_\psi. \quad (10)$$

Many mathematicians paid much attention to Hankel operators and Toeplitz operators. The formulas (7)–(10) show the close relationships among the above four kinds of operators, so it is reasonable to focus on the dual Toeplitz operators. In the setting of the Bergman space, algebraic and spectral properties of dual Toeplitz operators were studied in [1–5].

Hankel operators and Toeplitz operators on Dirichlet space have been studied as in [6–10]. In this paper, we investigate the dual Toeplitz operators on the orthogonal complement of the Dirichlet space. The general problem that we are interested in is the following: What is the relationship between their symbols when two dual Toeplitz operators commute? For