A Combinatorial Identity Arising from Symplectic Geometry

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Abstract In this note, we give a generalization of the famous combinational identity \((-1)^nn! = \sum_{k=1}^{n} \binom{n}{k}(-1)^kk^n\) arising from symplectic geometry.

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1 Introduction and Main Result

In this note we shall give a generalization of following famous combinational identity due to Boole (see [1])
\((-1)^nn! = \sum_{k=1}^{n} \binom{n}{k}(-1)^kk^n\) \hspace{1cm} (1)

by using techniques from symplectic geometry. Namely, we shall prove

Theorem 1.1 Let \(n, k\) and \(m = n - k\) be positive integers. Then
\[ (km)!4km(k-1)!(k-2)!\cdots1!(n-k-1)!\cdots2!1! \]
\[ = (-1)^km \sum_{0 \leq r \leq k} \sum_{P} \frac{[m(k + n + 1) - k(k + 1) - 2C(n, k) - 4 \sum_{s=1}^{r} (j_s - i_s)(j_s - l_\alpha)(l_\alpha - i_t)(l_\alpha - l_\beta)]^{km}}{\prod_{i=1}^{r} \prod_{s=1}^{k-r} \prod_{\beta=0}^{k-r+1} (j_s - i_t)(j_s - l_\beta)(l_\alpha - i_t)(l_\alpha - l_\beta)}, \] \hspace{1cm} (2)

where the subscript \(P\) in the second sum denotes all possible permutations \(1 \leq i_1 < i_2 < \cdots < i_r \leq k < k + 1 \leq j_1 < \cdots < j_r \leq n\) and their complements \(1 \leq l_1 < l_2 < \cdots < l_{k-r} \leq k < k + 1 \leq l_{k-r+1} < \cdots < l_{n-2r} \leq n\) in \(\{1, \ldots, n\}\) such that identity (24) is satisfied, and that \(C(n, k)\) is a constant determined by (19). It’s a convention that any product term in denominators of fractions on the right side of (2) is omitted directly while 0 appearing in its subscripts. As an example let us consider the case when \(r = k\). Then all terms having \(\alpha\) as subscripts are omitted. In other words, in this case only the terms \(j_s - i_t\) and \(j_s - l_\beta\) are retained in denominators.

Remark 1.2 In fact, the combinational identity (1) is a special case of Theorem 1.1 for \(k = 1\). We get
\[ C(n, 1) = \frac{n(n + 1)}{2} \]
from (20), then identity (1) follows directly from an easy calculation.

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Recall that identity (1) may be obtained from symplectic geometry, i.e., applying Duistermaat–Heckman theorem to the natural $S^1$ action on $\mathbb{C}P^n$, cf., [2]. Our generalization is obtained by applying Duistermaat–Heckman theorem to a suitable $S^1$ action on complex Grassmann manifolds. For the sake of completeness let us recall the precise content of the Duistermaat–Heckman theorem. A smooth vector field $X$ on $M$ is called a Hamiltonian vector field if each vector field $\xi \in \mathfrak{g}$ is a symplectic vector field on $M$, usually denoted by $\text{Symp}(M, \omega)$ or $\text{Symp}(M)$ for short, is an infinite-dimensional Lie group, cf., [3]. A smooth vector field $X$ on $M$ is called symplectic if $\iota_X \omega$ is closed. Moreover, if $\iota_X \omega$ is exact, then $X$ is called Hamiltonian. All symplectic (resp. Hamiltonian) vector fields on $M$, denoted by $\chi_{\text{Symp}}(M, \omega)$ (resp. $\chi_{\text{Ham}}(M, \omega)$), form a $C^\infty(M)$-module space. It is easy to see that

$$\chi_{\text{Ham}}(M, \omega) \subset \chi_{\text{Symp}}(M, \omega) \subset \chi(M, \omega),$$

where $\chi(M, \omega)$ is the set of all smooth vector fields on $M$. Assume that $M$ is closed. Then any smooth family of symplectic vector fields $\{X_t\}_{0 \leq t \leq 1}$ determines a flow of symplectomorphisms $\{\psi_t\}_{0 \leq t \leq 1}$ via equation $\frac{d}{dt} \psi_t = X_t \circ \psi_t$ with initial condition $\psi_0 = \text{id}$. Moreover, if all $X_t$ are Hamiltonian, then there exists a smooth family of Hamiltonian functions $H_t : M \to \mathbb{R}$ such that

$$\iota_{X_t} \omega = dH_t$$

for every $t \in [0, 1]$. In this case $H$ is called a time-dependent Hamiltonian function and $\{\psi_t\}_{0 \leq t \leq 1}$ is called a Hamiltonian isotropy on $(M, \omega)$. Every individual element $\psi_t$ is called a Hamiltonian diffeomorphism.

Let $G$ be a compact Lie group with Lie algebra $\mathfrak{g} = \text{Lie}(G)$ which acts on $(M, \omega)$ by symplectomorphisms. This means that there is a group homomorphism $\varphi : G \to \text{Symp}(M, \omega), g \mapsto \varphi_g$. In other words, $\varphi_g : M \to M$ is a symplectomorphism for every $g \in G$, and

$$\varphi_{gh} = \varphi_g \circ \varphi_h, \quad \varphi_e = \text{id},$$

for $g, h \in G$. Then infinitesimal action determines a homomorphism

$$\mathfrak{g} \to \chi(M, \omega), \xi \mapsto X_\xi \triangleq \left. \frac{d}{dt} \right|_{t=0} \varphi_{\exp(t\xi)}.$$

Since $\varphi_g$ is a symplectomorphism for every $g \in G$ it follows that each $X_\xi$ is a symplectic vector field. Moreover, $G$ is called a weak Hamiltonian action on $M$ if each vector field $X_\xi$ is Hamiltonian. This means that for any $\xi \in \mathfrak{g}$ there is a corresponding Hamiltonian function $H_\xi$ such that $\iota_\xi \omega = dH_\xi$. However, this function $H_\xi$ is only determined up to a constant. Obviously, we can choose appropriate constants such that the mapping

$$\mathfrak{g} \to C^\infty(M), \quad \xi \mapsto H_\xi$$

is linear. The action $G$ is called Hamiltonian if (3) is a Lie algebra homomorphism, that is,

$$H_{[\xi, \eta]} = \{H_\xi, H_\eta\}$$

for any $\xi, \eta \in \mathfrak{g}$.