Boundedness of the Anisotropic Maximal and Anisotropic Singular Integral Operators in Generalized Morrey Spaces

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Abstract In this paper we give the conditions on the pair \((\omega_1,\omega_2)\) which ensures the boundedness of the anisotropic maximal operator and anisotropic singular integral operators from one generalized Morrey space \(M_{p,\omega_1}\) to another \(M_{p,\omega_2}\), \(1 < p < \infty\), and from the space \(M_{1,\omega_1}\) to the weak space \(W\!\!M_{1,\omega_2}\).

Keywords Generalized Morrey spaces, anisotropic maximal operator, Hardy operator, anisotropic singular integral operator

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1 Introduction and main results

The theory of boundedness of classical operators of real analysis, such as maximal operator and singular integral operators etc, from one weighted Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with weighted Lebesgue spaces, general Morrey-type spaces also play an important role.

Let \(\alpha = (\alpha_1, \ldots, \alpha_n)\), \(\alpha_i \geq 1\), \(i = 1, \ldots, n\), \(|\alpha| = \sum_{i=1}^{n} \alpha_i\) and \(t^\alpha x \equiv (t^{\alpha_1}x_1, \ldots, t^{\alpha_n}x_n)\). Following [1, 2], the function \(F(x, \rho) = \sum_{i=1}^{n} x_i^2 \rho^{-2\alpha_i}\), considered for any fixed \(x \in \mathbb{R}^n\), is a decreasing one with respect to \(\rho > 0\) and the equation \(F(x, \rho) = 1\) is uniquely solvable in \(\rho(x)\). It is a simple matter to check that \(\rho(x-y)\) defines a distance between any two points \(x, y \in \mathbb{R}^n\). Thus \(\mathbb{R}^n\), endowed with the metric \(\rho\), results in a homogeneous metric space (see [1–3]). The balls with respect to \(\rho(x)\), centered at the origin and of radius \(r\) are simply the ellipsoids

\[
E_r(0) = \left\{ x \in \mathbb{R}^n : \frac{x_1^2}{r^{2\alpha_1}} + \cdots + \frac{x_n^2}{r^{2\alpha_n}} < 1 \right\},
\]

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with Lebesgue measure $|E_r(0)| = C(n)r^{1/|\alpha|}$. It is easy to see that $E_1(0) \equiv \mathbb{S}^{n-1}$ with respect to the Euclidean one.

Let $f \in L^{1,\infty}_c(\mathbb{R}^n)$. The anisotropic maximal function $Mf$ is defined by

$$Mf(x) = \sup_{t > 0} \frac{1}{|E(x,t)|} \int_{E(x,t)} |f(y)|dy,$$

where $|E(x,t)|$ is the Lebesgue measure of the ellipsoid $E(x,t)$ centered at $x$.

The boundedness of the maximal operator $M$ in Morrey spaces $\mathcal{M}_{p,\lambda}$ was proven in [4] (isotropic case) and in generalized Morrey spaces $\mathcal{M}_{p,\omega}$, $p \in (1, \infty)$ with a function $\omega(x,r)$ satisfying suitable doubling and integral conditions $\tilde{Z}_{p,|\alpha|}$ (see Section 2) in [5]. In more general substations, namely in local and global Morrey type spaces, the boundedness of the maximal operator $M$ has been investigated in [6–14].

**Definition 1.1** The function $k(x;\xi) : \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) \rightarrow \mathbb{R}$ is called a variable Calderón–Zygmund type kernel with mixed homogeneity if

i) For every fixed $x$ the function $k(x;\cdot)$ is a constant kernel satisfying:

1.a) $k(x;\cdot) \in C^\infty(\mathbb{R}^n \backslash \{0\})$;

b) $k(x;it^\alpha \xi) = \mu^{-|\alpha|}k(x;\xi)$, $t > 0$;

c) $\int_{S^{n-1}} k(x;\xi)d\sigma_\xi = 0$ and $\int_{S^{n-1}} |k(x;\xi)|d\sigma_\xi < \infty$;

ii) For every multiindex $\beta$, the inequality $\sup_{\xi \in \mathbb{S}^{n-1}} |D_\xi^\beta k(x;\xi)| \leq C(\beta)$ is satisfied independently of $x$.

Note that in the isotropic case $\alpha_i = 1$, $i = 1, \ldots, n$ and thus $|\alpha| = n$, Definition 1.1 gives rise to the classical Calderón–Zygmund kernels (see, for example, [15] and [16]). One more example is when $\alpha_1 = \cdots = \alpha_{n-1} = 1$, $\alpha_n = \overline{\alpha} \geq 1$. In this case we obtain the parabolic kernels studied by Jones in [17] and discussed in [2].

We consider the following anisotropic singular integral

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} k(x; x-y)f(y)dy$$

with a variable Calderón–Zygmund type kernel $k(x;\xi)$, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \backslash \{0\}$, satisfying a mixed homogeneity condition i). The boundedness of the operator $T$ in $L^p(\mathbb{R}^n)$, $p \in (1, \infty)$ was proven in [1, 2] and in Morrey spaces $\mathcal{M}_{p,\lambda}$ in [18] (isotropic case). The boundedness of the operator $T$ in generalized Morrey spaces $\mathcal{M}_{p,\omega}$, $p \in (1, \infty)$ with a function $\omega(x,r)$ satisfying suitable doubling and integral conditions $\tilde{Z}_{p,|\alpha|}$ in [19] (isotropic case in [5]), and the boundedness of the operator $T$ from $\mathcal{M}_{p,\omega_1}$ to $\mathcal{M}_{p,\omega_2}$, $1 < p < \infty$ satisfying integral conditions $(\omega_1,\omega_2) \in \tilde{Z}_{p,|\alpha|}$ were proven in [12, 13]. Our goal is to extend results in [6, 12–14] with a pair $(\omega_1,\omega_2)$ satisfying more large integral conditions $Z_{p,|\alpha|}$. In [7–13] the boundedness of the singular integral operators in local and global Morrey-type spaces has been investigated. Note that the global Morrey-type space is a more general space than the generalized Morrey space.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant $C$ independent of the appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that $A$ and $B$ are equivalent.

### 2 Generalized Morrey Spaces and Preliminary Results

Morrey spaces $\mathcal{M}_{p,\lambda}$ were introduced by Morrey in 1938 [20] and defined as follows: