(Co)Homology and Universal Central Extension of Hom-Leibniz Algebras

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Abstract Hom-Leibniz algebra is a natural generalization of Leibniz algebras and Hom-Lie algebras. In this paper, we develop some structure theory (such as (co)homology groups, universal central extensions) of Hom-Leibniz algebras based on some works of Loday and Pirashvili.

Keywords Hom-Leibniz algebra, (co)homology theory, central extension

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1 Introduction

In 1990s, Loday introduced a non-anticommutative version of Lie algebras whose brackets satisfy the Leibniz identities rather than Jacobi identities, therefore is called Leibniz algebra [1]. Recently, Hom-Lie algebras were introduced by Hartwig, Larson and Silvestrov in [2] as a part of a study of deformations of the Witt and the Virasoro algebras. A Hom-Lie algebra is a triple \((L, [\cdot, \cdot], \alpha)\), in which \(L\) is a vector space, \(\alpha\) is an endomorphism of \(L\), and the skew-symmetric bracket satisfies an \(\alpha\)-twisted variant of the Jacobi identity. Lie algebras are special cases of Hom-Lie algebras in which \(\alpha\) is the identity map. Makhlouf and Silvestrov introduced the notion of a Hom-Leibniz algebra \((L, [\cdot, \cdot], \alpha)\) in [3], which is a natural generalization of Liebniz algebras and Hom-Lie algebras. Leibniz algebra is a special case of Hom-Leibniz algebra in which \(\alpha\) is the identity map and Hom-Lie algebra is an anti-commutative version of Hom-Leibniz algebra [4–7].
In this paper, we shall develop some structure theory (such as (co)homology groups, central extensions) of Hom-Leibniz algebras based on some works in [1]. The rest of this paper is organized as follows. In the next section, basic definitions about Hom-dialgebras and Hom-Leibniz algebras are recalled. We define the notion of representation (module) of a Hom-Leibniz algebra and show that the representation of Hom-dialgebras gives rise to the representation of Hom-Leibniz algebras via an special bracket. This enables us to construct its (co)homology and central extension with nontrivial coefficients. In Section 3, we construct the chain complex $C^\alpha_{\ast}(L, V)$ and define the homology $H^\alpha_{\ast}(L, V)$ of Hom-Leibniz algebra $(L, \alpha)$ with coefficients in its representation $(V, \alpha_V)$. In Section 4, abelian extensions of Hom-Leibniz algebras are considered. In Section 5, we define the cohomology $H^\ast_{\ast}(L, V)$ of Hom-Leibniz algebra which plays an important role in the central extension of Hom-Leibniz algebras. In Section 6, we mainly discuss the universal central extension of Hom-Leibniz algebra and give some results analogous to group or Lie algebra theorem. Finally, we show that a nontrivial Hom-Leibniz analog of the $q$-deformed Virasoro algebra and Heisenberg–Virasoro algebra does not exist.

2 Representations of Hom-Leibniz Algebra

As a natural generalization of Leibniz algebra and Hom-Lie algebra, Makhlouf and Silvestrov define the Hom-Leibniz algebra in [3].

Definition 2.1 A Hom-Leibniz algebra is a triple $(L, [\cdot, \cdot], \alpha)$ consisting of a linear space $L$, bilinear map $[\cdot, \cdot]: L \times L \to L$ and a linear space homomorphism $\alpha: L \to L$ satisfying the Hom-Leibniz identity:

$$[[x, y], \alpha(z)] = [[\alpha(x), [y, z]], \alpha(y)] + [[x, z], \alpha(y)]$$

for all $x, y, z \in L$. (2.1)

By a homomorphism of Hom-Leibniz algebras $\varphi: (L_1, \alpha_1) \to (L_2, \alpha_2)$ we mean an algebra homomorphism from $L_1$ to $L_2$ such that $\varphi \alpha_1 = \alpha_2 \varphi$.

This is in fact a right Hom-Leibniz algebra. The dual notion of left Hom-Leibniz algebra is made out of the dual relation $[[x, y], \alpha(z)] = [[\alpha(x), [y, z]], \alpha(y)] - [[x, z], \alpha(y)]$ for all $x, y, z \in L$. In this paper, we are considering only right Hom-Leibniz algebras. Denote by $H\text{Leib}$ the category of Hom-Leibniz algebras over $\mathbb{F}$. If $\alpha = Id_L$, then a Hom-Leibniz algebra becomes a Leibniz algebra. A Hom-Leibniz algebra is a Hom-Leibniz algebra. A Hom-Leibniz algebra is a Hom-Lie algebra if and only if $[x, x] = 0$, $\forall x \in L$. With any Hom-Leibniz algebra $(g, \alpha_g)$ there is associated a Hom-Lie algebra $(g_{\text{Lie}}, \alpha_{g_{\text{Lie}}})$, obtained by quotienting by the antisymmetry relation.

Suppose that $(L, \alpha)$ is a Hom-Leibniz algebra. For any $x \in L$, we define $Ad_x \in End_{\mathbb{F}}L$ by $Ad_x(y) = [x, y]$, for any $y \in L$. Then the Hom-Leibniz identity (2.1) can be written as

$$Ad_{\alpha(z)}([x, y]) = [\alpha(x), Ad_z(y)] + [Ad_z(x), \alpha(y)],$$

or in pure operator form

$$Ad_{\alpha(z)} \circ Ad_y = Ad_{Ad_z(y)} \circ \alpha + Ad_{\alpha(y)} \circ Ad_z.$$ (2.3)

Let $(L, \alpha)$ be a Hom-Leibniz algebra. A subalgebra $(S, \alpha|_S)$ of $(L, \alpha)$ is an $\mathbb{F}$-subspace which is closed under the bracket $[\cdot, \cdot]$, that is, $[S, S] \subseteq S$. We call $(I, \alpha|_I)$ a two-sided ideal if