Convex Mean Curvature Flow with a Forcing Term in Direction of the Position Vector

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Abstract A smooth, compact and strictly convex hypersurface evolving in $\mathbb{R}^{n+1}$ along its mean curvature vector plus a forcing term in the direction of its position vector is studied in this paper. We show that the convexity is preserving as the case of mean curvature flow, and the evolving convex hypersurfaces may shrink to a point in finite time if the forcing term is small, or exist for all time and expand to infinity if it is large enough. The flow can converge to a round sphere if the forcing term satisfies suitable conditions which will be given in the paper. Long-time existence and convergence of normalization of the flow are also investigated.

Keywords Evolution equation, mean curvature flow, forcing term, normalization

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1 Introduction

Let $M_0$ be a compact, strictly convex hypersurface of dimension $n \geq 2$, without boundary, smoothly embedded in $\mathbb{R}^{n+1}$ and represented by some diffeomorphism $X_0 : \mathbb{R}^n \supset U \to X_0(U) \subset M_0 \subset \mathbb{R}^{n+1}$. The forced mean curvature flow is a smooth family of maps $X_t = X(\cdot, t)$ evolving according to

$$\begin{cases}
\frac{d}{dt} X(x, t) = (h(t) - H(x, t))v(x, t), & x \in M^n, \ t > 0, \\
X(\cdot, 0) = X_0,
\end{cases}$$

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where $H$ is the mean curvature of $M_t = X_t(M^n)$, and $h(t)$ is a continuous function, and $v(x,t)$ the outer unit normal vector of $M_t$ at $X(x,t)$.

When $h(t)$ is non-negative, (1.1) is a contractive flow for $h$ small, an expanding flow for $h$ large enough, and (1.1) can converge to a round sphere for suitable $h$ (cf. [1]). Specially, (1.1) is the well-known mean curvature flow for $h = 0$ [2]. When $h(t) = \int_{M_t} HE_{k+1} d\mu / \int_{M_t} E_{k+1} d\mu$, $k = -1, 0, 1, \ldots, n - 1$, where $E_l$ is the $l$-th elementary symmetric function of the principal curvatures of $M$, (1.1) is the mixed volume preserving mean curvature flow [3], which includes the volume preserving case [4] and surfaces area preserving case [5]. For curvature flow with an external force field, we refer to [6, 7].

The main theorem we prove is

**Theorem 1.1** Let $M_0$ be an $n$-dimensional smooth, compact and strictly convex hypersurface immersed in $\mathbb{R}^{n+1}$ with $n \geq 2$. Then for any continuous function $\kappa(t)$, there exists a unique,