Chromatic Sums of Nonseparable Near-Triangulations on the Projective Plane

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Abstract In this paper, we study the chromatic sum functions of rooted nonseparable near-triangulations on the sphere and the projective plane. The chromatic sum function equations of such maps are obtained. From the chromatic sum equations of such maps, the enumerating function equations of such maps are derived. Applying chromatic sum theory, the enumerating problem of different sorts maps can be studied, and a new method of enumeration can be obtained. Moreover, an asymptotic evaluation and some explicit expression of enumerating functions are also derived.

Keywords Triangulation, chromatic sum function, enumerating function, asymptotic evaluation

MR(2000) Subject Classification 05C10, 05C30

1 Introduction

The chromatic polynomials have been studied by Loerinc [1], Read and Whitehead Jr. [2]. On chromatic sums, the first paper which was published in 1973 by Tutte [3] is for rooted planar triangulations. Since 1973, Tutte has published a series of papers [4–7] on chromatic sums for rooted planar triangulations to tackle the coloring average problem. About ten years later, Liu [8, 9] has studied the chromatic sum of rooted nonseparable maps on the plane. All results of chromatic sums having been published are on the plane. Chromatic sums of nonplanar maps are more difficult to be determined. And they are more general and more difficult than enumeration, because of the occurrence of chromatic polynomials. Chromatic sums play an important role in the study of some coloring average.

A surface is a compact 2-manifold. An (A) orientable (non-orientable) surface of genus $g$ is homeomorphic to the sphere with $g$ handle (or crosscaps) and is denoted by $S_g$ (or $N_g$). A map $M$ on (or embedded on) $S_g$ (or $N_g$) is a graph drawn on the surface so that each vertex is a point on the surface, each edge $\{x, y\}$, $x \neq y$, is a simple open curve whose endpoints are $x$ and $y$, each loop incident to a vertex $x$ is a simple closed curve containing $x$, no edge contains a vertex to which it is not incident, and each connected region of the complement of the graph in the surface is homeomorphic to a disc and is called a face. A map is rooted if an edge, a direction along the edge, and a side of the edge are all distinguished. If the root is the oriented edge from $u$ to $v$, then $u$ is the root-vertex while the face on the oriented side of the edge is defined as the root-face.

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A map always denoted by $M = (\mathcal{X}; \mathcal{J})$, where $\mathcal{X} = \sum_{x \in \mathcal{X}} \mathcal{K}x$, $\mathcal{K}x = \{x, \alpha x, \beta x, \alpha \beta x\}$, $\mathcal{K}$ is the Klein group of four elements denoted by $1, \alpha, \beta, \alpha \beta$, $\mathcal{X}$ is a finite set, and $\mathcal{J}$ is a basic permutation on $\mathcal{X}$. Terms not mentioned here can be found in [10].

1.1 Figure

A (rooted) near-triangular map on a surface is a (rooted) map on the surface, such that each face except possibly the root face has the valency three. A (rooted) triangular map is a (rooted) near-triangular map with root face valency also being three. A map $M$ is called separable if its edge set can be partitioned into two disjoint non-null submaps $S$ and $T$ so that there is just one vertex incident with both $S$ and $T$, the vertex is said to be a separable vertex of $M$. A rooted nonseparable near-triangular map is a rooted near-triangular map without any separable vertex. A circuit $C$ on a surface $\Sigma$ is called essential, if $\Sigma - C$ has no connected region homeomorphic to a disc, otherwise it is planar. An edge is called double edge (a double edge on the plane is also called isthmus by some authors) if each side of it is on the boundary of the same face.

Ren [11] has studied the enumeration of rooted nonseparable near-triangulations on the projective plane with the root-face valency, the number of edges and the number of inner faces. Let $\mathcal{S}$ and $\mathcal{M}$ be respectively the set of all rooted nonseparable near-triangulations on the sphere and the projective plane. Their chromatic sum functions are, respectively,

$$f = f(x, y, z, t, \omega; \lambda) = \sum_{M \in \mathcal{S}} P(M; \lambda)x^{m(M)}y^{n(M)}z^{l(M)}t^{s(M)}\omega^{d(M)};$$

$$F = F(x, y, z, t, \omega; \lambda) = \sum_{M \in \mathcal{M}} P(M; \lambda)x^{m(M)}y^{n(M)}z^{l(M)}t^{s(M)}\omega^{d(M)},$$

where $m(M)$, $n(M)$, $l(M)$, $s(M)$ and $d(M)$ are respectively the valency of the root-vertex of $M$, the valency of the root-face of $M$, the number of edges of $M$, the number of nonroot-vertices of $M$ and the number of nonroot-faces of $M$. $P(M; \lambda)$ is the chromatic polynomial of $M$.

Now two well-known formula on chromatic polynomials of maps should be mentioned for further use. The first one is

$$P(M; \lambda) = P(M - e; \lambda) - P(M \bullet e; \lambda).$$

(1.1)

Figure 1