Real Interpolation between Martingale Hardy and BMO Spaces

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Abstract In this paper, we consider the real interpolation with a function parameter between martingale Hardy and BMO spaces. An interpolation theorem for martingale Hardy and BMO spaces is formulated. As an application, real interpolation between martingale Lorentz and BMO spaces is given.

Keywords Martingale space, BMO space, Lorentz space, real interpolation, function parameter

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1 Introduction

The real interpolation spaces $A_{\theta,q}$ were introduced in [1]. The theory for these spaces has been applied as a powerful tool in many branches of mathematics. The spaces $A_{\theta,q}$ are defined by using the function norm:

$$\phi_{\theta,q}(\varphi) = \left( \int_0^{\infty} (t^{-\theta} \varphi(t))^{q} \frac{dt}{t} \right)^{1/q}.$$

For the further applications of interpolation space theory, the idea of replacing $\phi_{\theta,q}$ by a more general function norm to obtain more general interpolation spaces appeared. The initial work on such spaces was due to Kalugina [2]. He used the function norm

$$\phi_{f,p}(\varphi) = \left( \int_0^{\infty} (\varphi(t)/f(t))^{p} \frac{dt}{t} \right)^{1/p}$$

to replace $\phi_{\theta,q}$, where $f$ is a function parameter, and belongs to the function class $B_K$. Later on the theory of interpolation with a function parameter has been developed in an astounding way. For example, see [3–7]. In particular, we mention the work of Persson [8], in which interpolation with a function parameter belonging to the function class $Q(0,1)$ was systematically investigated.

The interpolation spaces between the classical Hardy and BMO spaces were identified by Hanks [9] and Bennett, Sharpley [10]:

$$(H_{p_0,q_0}, \text{BMO})_{\theta,q} = H_{p,q}, \quad \frac{1}{p} = \frac{1 - \theta}{p_0}, \quad 0 < \theta < 1, \quad 0 < p_0 < \infty, \quad 0 < q_0, q \leq \infty.$$
It is one of important results in classical interpolation theory. A similar result for martingale Hardy–Lorentz and BMO spaces was proved by Weisz [11]. As an important part of martingale $H_p$ theory, interpolation theory of martingale spaces has attracted more attentions in recent years. Many literatures about this have been published one after another. For example, see [12–15].

The purpose of this paper is to consider real interpolation with a function parameter between martingale Hardy and BMO spaces. In this paper, we apply the function parameter introduced by Persson [8] to real interpolation for martingale Hardy and BMO spaces. An interpolation theorem between martingale Hardy and BMO spaces is formulated. As an application, real interpolation between martingale Lorentz and BMO spaces is given.

The organization of this paper is divided into three further sections. Some basic knowledge, which we will use, is collected in the next section. In Section 3 some lemmas are provided. Main results and proofs are given in Section 4.

2 Preliminaries

Let $(X, \mu)$ be a $\sigma$-finite measure space, $\mathcal{M}(X)$ the space of all measurable functions on $X$. For $f \in \mathcal{M}(X)$, denote its distribution function by

$$\lambda_f(t) = \mu(x : |f(x)| > t), \quad t \geq 0,$$

and its decreasing rearrangement function $f^*$ is defined as

$$f^*(t) = \inf \{ s > 0 : \lambda_f(s) \leq t \}, \quad t \geq 0.$$

For $0 < q \leq \infty$, let $\varphi$ be a nonnegative and locally integrable function on $[0, \infty)$ ($\varphi \neq 0$), the classical Lorentz spaces are defined as

$$\Lambda_q(\varphi) = \{ f \in \mathcal{M}(X) : \| f \|_{\Lambda_q(\varphi)} < \infty \},$$

where

$$\| f \|_{\Lambda_q(\varphi)} = \begin{cases} \left( \int_0^\infty (f^*(t)\varphi(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}, & q < \infty, \\ \sup_{t>0} f^*(t)\varphi(t), & q = \infty. \end{cases}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\{\mathcal{F}_n\}_{n \geq 0}$ a nondecreasing sequence of sub-$\sigma$-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. The conditional expectation operators relative to $\mathcal{F}_n$ are denoted by $\mathbb{E}_n$. For a martingale $f = (f_n)_{n \geq 0}$ relative to $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_n)_{n \geq 0})$, denote its martingale differences by $df_i = f_i - f_{i-1}$ ($i \geq 0$, with convention $df_0 = 0$) and its conditional quadratic variation by

$$s_n(f) = \left( \sum_{i=1}^n \mathbb{E}_{i-1}[|df_i|^2] \right)^{\frac{1}{2}}, \quad s(f) = \left( \sum_{i=1}^\infty \mathbb{E}_{i-1}[|df_i|^2] \right)^{\frac{1}{2}}.$$

The sharp function $f_r^*$ of a martingale $f = (f_n)_{n \geq 0}$ is defined as

$$f_r^* = \sup_{n \geq 0} \eta_n = \sup_{n \geq 0} (\mathbb{E}_n[s^2(f) - s_n^2(f)]^\frac{1}{2}), \quad 0 < r < \infty.$$