A Bishop–Stone–Weierstrass Theorem for \((\mathcal{M}_2(\mathbb{C}))^n\)

Qi Hui LI  
Department of Mathematics, East China University of Science and Technology,  
Shanghai 200237, P. R. China  
E-mail: lqh991978@gmail.com

Don HADWIN  
Department of Mathematics and Statistics, University of New Hampshire,  
Durham, NH 03824, USA  
E-mail: operatorguy@gmail.com

Abstract  In this paper, we shall give an elementary proof of a Bishop–Stone–Weierstrass theorem for \((\mathcal{M}_2(\mathbb{C}))^n\) with respect to its pure states. To be more precise, we shall show that the pure-state Bishop hull of a unital subalgebra (not necessarily self-adjoint) of \((\mathcal{M}_2(\mathbb{C}))^n\) is equal to itself.

Keywords  Bishop–Stone–Weierstrass theorem, pure states, C*-algebras

MR(2010) Subject Classification  15A60, 47L40

1 Introduction  
The problem of extending the Stone–Weierstrass theorem to C*-algebras has been studied for a great many years [1, 2, 4–6, 8–10, 12–16]. Consider a compact Hausdorff space \(X\), and the set \(C(X)\) of all continuous complex-valued functions on \(X\). The classical Stone–Weierstrass theorem states that if \(A\) is a closed self-adjoint unital subalgebra of \(C(X)\) and contains sufficiently many functions to distinguish points of \(X\), then \(A = C(X)\). To extend the Stone–Weierstrass theorem into the framework of the general C*-algebra theory, the basic idea is to let \(X\) be a subset of the set of states on a C*-algebra \(A\) and to identify \(A\) as a linear subspace of \(C(X)\). The Stone–Weierstrass problem for \(A\) and \(X\) is whether each unital C*-subalgebra \(B\) of \(A\) that separates the points of \(X\) must be \(A\). An affirmative answer was given by Glimm [9] in the case in which \(X\) is the weak*-closure of the set of pure states on \(A\). Popa [14] and Longo [13] independently gave an affirmative answer for the case in which \(A\) is separable and \(X\) is the set of factor states on \(A\). Although much progress has been made, the case in which \(X\) is the set of pure states on \(A\) is still unsolved.

The following classical result is Bishop’s generalization of the Stone–Weierstrass theorem. Recall that a non-empty subset \(S\) of \(X\) is said to be \(A\)-antisymmetric if whenever \(h \in A\) and \(h\) is real-valued on \(S\), then \(h\) is constant on \(S\).

**Theorem 1.1** ([3, Bishop’s theorem])  
Suppose that \(A\) is closed in \(C(X)\). If \(f \in C(X)\), and if for each \(A\)-antisymmetric subset \(S\) of \(X\) there exists \(g \in A\) such that \(f|_S = g|_S\), then \(f \in A\).
In this paper, we are going to consider C*-algebraic generalization of Bishop’s extension of the Stone–Weierstrass theorem for \((M_2(\mathbb{C}))^n\).

One important feature of Bishop’s theorem is that it tells when an element of \(C(X)\) is in a given algebra. How do we extend Bishop’s theorem into the framework of general C*-algebra theory? Suppose that \(A\) is a unital C*-algebra, \(B\) is a unital norm closed (not necessarily selfadjoint) subalgebra of \(A\) and \(S(A)\) is the set of states on \(A\). If we give \(S(A)\) the weak*-topology, we can identify \(A\) with a norm closed linear subspace of \(C(S(A))\). Thus, for \(a \in A\) and \(f \in S(A)\), we can use the notation \(f(a)\) and \(a(f)\) interchangeably. More often, if \(E \subseteq S(A)\) and \(a \in A\), we let \(a|_E\) denote the restriction of \(a\) to \(E\), let \(a(E)\) denote \(\{a(f): f \in E\}\), and let \(\mathcal{B}|_E\) denote \(\{b|_E: b \in \mathcal{B}\}\).

A subset \(E\) of \(S(A)\) is \(\mathcal{B}\)-antisymmetric if, for every \(b \in \mathcal{B}\), the restriction \(b|_E\) is constant whenever \(b|_E\) is real. Suppose \(X \subseteq S(A)\). We define the Bishop hull \(\text{Bish}(\mathcal{B}, X)\) of the algebra \(\mathcal{B}\) with respect to \(X\) by

\[
\text{Bish}(\mathcal{B}, X) = \{a \in A: a|_E \in \mathcal{B}|_E \text{ for each } \mathcal{B}\text{-antisymmetric subset } E \text{ of } X\}.
\]

Throughout this paper, we will denote the set of all pure states on \(A\) by \(\mathcal{P}(A)\). In [10], Hadwin investigated the following conjecture.

**Conjecture** Suppose \(A\) is a unital C*-algebra and \(\mathcal{B}\) is a unital subalgebra of \(A\). Then \(\mathcal{B} = \text{Bish}(\mathcal{B}, \mathcal{P}(A))\).

Although the conjecture has not been proved yet, much progress was made in [10]. The following theorem is one of main results in [10]. It provides a true generalization of the Bishop–Stone–Weierstrass theorem on \(C(X)\).

**Theorem 1.2** ([10, Theorem 4.2]) Suppose \(A\) is a unital C*-algebra and \(\mathcal{B}\) is a separable unital subalgebra of \(A\) which is commutative. Then \(\mathcal{B} = \text{Bish}(\mathcal{B}, \mathcal{P}(A))\).

In this article, we shall give a proof of the conjecture when \(A = (M_2(\mathbb{C}))^n\). Even though, we did not give the answer to the whole problem; our main result gives an affirmative answer for a non-commutative case. The proof is not as easy as one thought.

A brief overview of this paper is as follows. For the sake of completeness, in Section 2, we recall some basic properties of C*-algebras. Section 3 is devoted to some results on the direct sum of C*-algebras. In Sections 4 and 5, we fix our notation and find out all unital subalgebras of \(M_2(\mathbb{C})\) and \(M_2(\mathbb{C}) \oplus M_2(\mathbb{C})\) respectively. We also give elementary proofs of the Bishop–Stone–Weierstrass theorem for \(M_2(\mathbb{C})\) and \(M_2(\mathbb{C}) \oplus M_2(\mathbb{C})\) respectively, with respect to the pure states. Section 6 is dedicated to the main result of this paper. Based on the proofs in Sections 4 and 5, we will give an elementary proof of the Bishop–Stone–Weierstrass theorem for \((M_2(\mathbb{C}))^n\) with respect to its pure states.

## 2 Preliminaries

Suppose that \(A\) is a C*-algebra. The following three lemmas are well-known, so we shall omit their proofs.

**Lemma 2.1** ([7, Corollary III.1.2]) If \(\pi\) is a non-degenerated representation of a finite-dimensional C*-algebra \(A = M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_n}(\mathbb{C})\), then there are cardinal numbers \(\alpha_1, \ldots, \alpha_k\) such that \(\pi\) is unitary equivalent to \(\text{id}_{1^{(\alpha_1)}}^1 \oplus \text{id}_{2^{(\alpha_2)}}^2 \oplus \cdots \oplus \text{id}_{n^{(\alpha_n)}}^n\) where \(\text{id}_j\) denotes the identity representation of \(M_{n_j}(\mathbb{C})\).