Non-differentiability of Alpha Function at the Boundary of Flat

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Abstract With the variational method introduced by Mather, we construct a mechanical Hamiltonian system whose \( \alpha \) function has a flat \( \mathcal{F} \) and is non-differentiable at the boundary \( \partial \mathcal{F} \). In the case of two degrees of freedom, we prove that this phenomenon is stable under perturbations of Mañé’s.

Keywords Mather theory, \( \alpha \) function, mechanical systems

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1 Introduction

Let \( M \) be a smooth closed manifold with \( TM \) as tangent bundle. We call such a function \( L(x, v) \in C^r(TM, \mathbb{R}) \) \( r \geq 2 \) Tonelli Lagrangian if it satisfies:

- **convexity:** For all \( x \in M, v \in T_x M \) the Hessian matrix \( \frac{\partial^2 L}{\partial v_i \partial v_j}(x, v) \) is positive definiteness;
- **superlinearity:** \( \lim_{\|v\| \to \infty} \frac{L(x, v)}{\|v\|} = \infty \) uniformly on \( (x, v) \in TM \);
- **completeness:** All solutions of the corresponding Euler–Lagrangian equation are well-defined for \( t \in \mathbb{R} \), here the Euler–Lagrangian equation (E–L) is given by
  \[
  \frac{d}{dt} \frac{\partial L}{\partial v}(x, v) - \frac{\partial L}{\partial x}(x, v) = 0, \quad (x, v) \in TM. \tag{1.1}
  \]

**Remark 1.1** In the autonomous case, the completeness is natural under the first two assumptions. That is because we can get the Hamiltonian via the Legendre Transformation

\[
H(x, v) = \frac{\partial L}{\partial v}(x, v) - L.
\]

From [5] we know that along each orbit \( (\gamma, \dot{\gamma}) \) of Euler–Lagrangian equation \( H(\gamma, \dot{\gamma}) \) is constant. The superlinearity implies that every level-set of Hamiltonian is compact. This in turn assures the completeness of flow.

Usually, we take \( M = \mathbb{T}^n \). Substracting a closed 1-form \( \eta_c \) with the cohomology class \( [\eta_c] = c \in H^1(M, \mathbb{R}) \), we get a new Tonelli Lagrangian \( L - \eta_c \) written by \( L - c \) for short.

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From [7], we know that the E-L flow \((\gamma, \dot{\gamma})\) of \(L - \eta_c\) also satisfies the E-L equation of \(L\). So we can define a \(c\)-minimal curve \(\gamma \in C^1(\mathbb{R}, M)\) if it satisfies:

\[
\mathcal{K}_c(\gamma) = \min_{\xi(s) = \gamma(s)} \int_a^b (L - \eta_c)(\xi(t), \dot{\xi}(t))dt, \quad \forall a < b \in \mathbb{R}, \, \xi \in C^{ac}(\mathbb{R}, M).
\]

All of the \(c\)-minimal orbit \((\gamma, \dot{\gamma})\) form a set denoted by \(\tilde{G}(c)\), which is invariant under the Euler–Lagrangian flow \(\Phi^c\).

Let \(\mathcal{M}_{inv}\) be the set of \(\Phi^c\)-invariant probability measures on \(TM\). We define the \(\alpha\) function as follows:

\[
\alpha(c) = -\min_{\mu \in \mathcal{M}_{inv}} \int_{TM} (L - \eta_c)d\mu, \quad c \in H^1(M, \mathbb{R}).
\]

As is showed in [7], there exists at least one measure \(\mu_c\) such that the minimum attains. We call this measure \(\mu_c\) \(c\)-minimal measure. The union of the supports of all \(c\)-minimal measures is called Mather set, denoted by \(\tilde{\mathcal{M}}(c)\). We call the minimal value of \(\alpha\) function Mañé Critical Value.

Since we know that \(\alpha(c)\) is convex, finite everywhere and superlinear [7], we can define its conjugate function in the sense of convex analysis [8] as

\[
\beta(h) = \min_{\mu \in \mathcal{M}_{inv}} \int_{TM} Ld\mu, \quad h \in H_1(M, \mathbb{R}),
\]

here \(\rho(\mu)\) is defined via the De Rham inner product:

\[
\langle \rho(\mu), c \rangle = \int \eta_c d\mu.
\]

\(\beta(h)\) is also a convex, finite everywhere and superlinear function. From [6] and [10] we can get the following properties:

- If \(\mu\) is a \(c\)-minimizing measure, we have \(\rho(\mu) \in D^- \alpha(c)\).
- The maximal connected domain on which \(\alpha\) function is not strict convex is called a flat \(F\). \(\forall c, c' \in \text{int} F\) we have \(\tilde{\mathcal{M}}(c) = \tilde{\mathcal{M}}(c')\).
- For each non-differential point \(c\) of \(\alpha\) function, \(\tilde{\mathcal{M}}(c)\) corresponds to at least two ergodic components with different rotation vectors.
- If \(h\) is a strict convex point of \(\beta\) function, then there must exist one ergodic minimal measure \(\mu\) with \(\rho(\mu) = h\).

We also need to define another two sets called Aubry set \(\tilde{A}(c)\) and Mañé set \(\tilde{N}(c)\). First, we define

\[
h_c^t(x, y) = \min_{\gamma \in C^1([0, t]; M)} \int_0^t ((L - \eta_c)(\gamma(s), \dot{\gamma}(s)) + \alpha(c))ds,
\]

\[
\Phi_c(x, y) = \inf_{t \in \mathbb{R}} h_c^t(x, y),
\]

and

\[
h_c^\infty(x, y) = \lim_{t \to +\infty} h_c^t(x, y).
\]

We call a curve \(\gamma \in C^1(\mathbb{R}, M)\) \(c\)-semi-static if

\[
\Phi_c(\gamma(a), \gamma(b)) = \int_a^b ((L - \eta_c)(\gamma(t), \dot{\gamma}(t)) + \alpha(c))dt, \quad \forall a < b \in \mathbb{R},
\]