Normal Functions and $\alpha$-Normal Functions

Yan Xu

Department of Mathematics, Nanjing Normal University, Nanjing 210097 P. R. China

Abstract This paper has studied two open questions about normal functions due to Lappan, and obtained two corresponding results for $\alpha$-normal functions.

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1 Introduction

Let $D$ denote the unit disc in the complex plane $\mathbb{C}$, and let $f$ be a meromorphic function in $D$. We call $f^\#$ defined by $f^\#(z) = |f(z)|/(1 + |f(z)|^2)$ the spherical derivative of $f$. Let $\overline{\mathbb{C}}$ denote the extended complex plane and let $\chi(z_1, z_2)$ denote the spherical distance between complex values $z_1, z_2 \in \overline{\mathbb{C}}$, that is $\chi(z_1, z_2) = \frac{|z_1 - z_2|}{(1 + |z_1|^2)^{1/2}(1 + |z_2|^2)^{1/2}}$. Then $f^\#(z) = \lim_{t \to z} \chi(f(t), f(z))$. So we have $\chi(f(z_1), f(z_2)) \leq \int_{L} f^\#(z) |dz|$, where $z_1$ and $z_2$ are two points in $D$ and $L$ is the line segment connecting $z_1$ and $z_2$.

A function $f$ meromorphic in $D$ is called a normal function if there exists a constant $c(f)$ such that $(1 - |z|^2)f^\#(z) \leq c(f)$ for each $z \in D$ (see [1]). Given $0 < \alpha < \infty$, if there exists a constant $c(f)$ such that $(1 - |z|^2)^\alpha f^\#(z) \leq c(f)$ for each $z \in D$, we say that $f$ is an $\alpha$-normal function in $D$ (see [2]).


**Theorem A** Let $f$ be a normal meromorphic function in $D$. Then for each positive integer $n$ there exists a constant $c_n(f)$ such that $(1 - |z|^2)^n \prod_{j=0}^{n-1} f^{(j)}(z) \leq c_n(f)$ for each $z \in D$.

For a meromorphic function $f$ in $D$ and a positive integer $n$, the expression $|f^{(n)}(z)|/(1 + |f(z)|^{n+1})$ represents an extension of the spherical derivative of $f$. As for this expression, Lappan has proved the following result:

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Theorem B [3] If \( f \) is a normal meromorphic function in \( D \), then for each positive integer \( n \) there exists a constant \( P_n(f) \) such that

\[
(1 - |z|^2)^n \frac{|f^{(n)}(z)|}{1 + |f(z)|^4} \leq P_n(f)
\]

for each \( z \in D \).

In [3], Lappan also suggested the following two open questions relating to Theorem A and Theorem B respectively.

**Question 1** Suppose that for a given meromorphic function \( f \) in \( D \) there exist a specific positive integer \( n_0 \) and a constant \( Q \) such that

\[
(1 - |z|^2)^{n_0} \prod_{k=0}^{n_0-1} (f^{(k)})^\#(z) \leq Q
\]

for each \( z \in D \). Must one of the functions \( f, f', \ldots, f^{(n_0)} \) be a normal function?

**Remark 1** Obviously, the answer for \( n_0 = 1 \) is positive.

**Question 2** If there exist \( n_0 > 1 \) and a constant \( P \) such that

\[
(1 - |z|^2)^{n_0} |f^{(n_0)}(z)| \leq P
\]

for each \( z \in D \), is \( f \) an normal function?

In this paper, we will show by example that the answer to Question 1 is negative for \( n_0 \geq 2 \). As for Question 2, we give a positive answer with the additional condition that all the zeros of \( f \) are of multiplicity at least \( n_0 \). In addition, we obtain two corresponding results for \( \alpha \)-normal functions.

2 Lemmas

To prove our results, we need some preliminaries.

**Lemma 1** [4] Let \( k \) be a positive integer and let \( f \) be a function meromorphic in \( D \) such that \( f \) has only zeros of multiplicity at least \( k \). If \( f \) is not a normal function, then, for each positive number \( \lambda < k \), there exist a sequence of points \( \{z_n\} \) in \( D \) such that \( |z_n| \to 1 \), and a sequence of positive numbers \( \{\rho_n\} \) such that \( \rho_n/(1 - |z_n|^2) \to 0 \), for which the sequence

\[
\{g_n(\zeta) = (1 - |z_n|^2)^{\lambda} \rho_n^{-\lambda} f(z_n + \rho_n \zeta)\}
\]

converges spherically and locally uniformly to a non-constant meromorphic function in the \( \zeta \)-plane.

The following result is due to Wulan [2]. For the convenience of the reader, here we give the proof of the sufficiency part. In fact, we make use of only the sufficiency part of this result in this paper.

**Lemma 2** Let \( \alpha \geq 1 \) and let \( f \) be a meromorphic function in \( D \). Then, \( f \) is an \( \alpha \)-normal function if and only if the family \( \{F_\alpha(z) = f(\phi_\alpha((1 - |a|^2)^{\alpha-1}z)) : a \in D\} \) is normal in \( D \), where \( \phi_\alpha(z) = (z + a)/(1 + \pi z) \).

**Proof of the sufficiency part** If the family is normal, then

\[
F_\alpha^\#(0) = f^\#(a)(1 - |a|^2)(1 - |a|^2)^{\alpha-1} = (1 - |a|^2)^\alpha f^\#(a) \leq M,
\]

where \( M \) is independent of \( a \). Thus \( f \) is \( \alpha \)-normal. The proof of the sufficiency part is completed.