Classification of Wavelet Bases by Translation Subgroups and Nonharmonic Wavelet Bases

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Abstract The structure of the set $S$ of shiftable points of wavelet subspaces is researched in this paper. We prove that $S = \mathbb{R}$ or $S = \frac{1}{q} \mathbb{Z}$ where $q \in \mathbb{N}$. The spectral and functional characterizations for the shiftability are given. Furthermore, the nonharmonic wavelet bases are discussed.

Keywords Wavelet, Translation invariance, Functional characterization

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1 Introduction

Wavelet analysis has been an attractive area of study in the past decade. Unfortunately the lack of shiftability of wavelet bases is a serious drawback for many applications such as image matching and communication, etc. (cf. [1–3]). There are few papers that deal with the shiftability problem in mathematics. In this paper we will discuss some aspects of it, the main idea and technique is built in the author’s Ph. D thesis [4], where a more general shiftability problem is researched with the unitary representative of local compact abelian groups under the background of wavefunction evolution in quantum mechanics.

Why do wavelet bases have the ability to locate the singularities of a function? This is because they damage the symmetry of the translates heavily, while the Fourier bases are stationary under translates. Formally speaking, if $a_n(s)$ is the Fourier coefficient of a $2\pi$ periodic function $f_s(x) = f(x - s)$, then the modular $|a_n(s)|$ is invariant. But in the wavelet expansion this kind of symmetry does not exist.

In this paper, we will discuss some aspects of the shiftability problem. In the next section we give the detailed classification and characterization for shiftability of principal shift invariant
spaces, and discuss the corresponding problem for wavelet spaces. In Sec. 3, we tend to the perturbation of wavelet bases relative to shiftability and required proof. All the wavelet bases discussed here are unconditional, and can be built with the multiresolution analysis model.

2 Shiftability of Principal Shift Invariant Space

We concentrate on the principal shift invariant spaces spanned by integer translates of single function, since both the scaling spaces and wavelet subspaces can be regarded as this type. Without loss of any essential content, we assume that the generator $\phi(x)$ spans the space $V$ orthonormally, i.e. $\langle \phi(x - m), \phi(x - n) \rangle = 1$ for $m = n$ and zero otherwise. An equivalent frequency form is

$$\sum_{n=-\infty}^{+\infty} |\hat{\phi}(\xi + 2n\pi)|^2 = 1, \quad \forall (a.e)\xi \in \mathbb{R},$$

(2.1)

where the Fourier transform is defined as $\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x) \exp(-i\pi x) dx$. We stress that

$$|\hat{\phi}(\xi)| \leq 1, \quad \forall (a.e)\xi \in \mathbb{R},$$

(2.2)

which is a consequence of (2.1). The shiftable set is defined as $S = \{ s \in \mathbb{R}; \phi(x - s) \in V \}$. An obvious fact is $\forall f(x) \in V, f(x-s) \in V$ if $s \in S$. Using the general plus “+” operation, this set $S$ forms a topological group $(S,+)$. Furthermore $(\mathbb{Z},+) \subset (S,+) \subset (\mathbb{R},+)$. We divide it into two cases for discussion purposes.

Case 1 $S$ is dense in $\mathbb{R}$.

The translates on $V$, $\tau_s: f(x) \mapsto f(x-s), f(x) \in V$, form a continuous unitary representative of topological group $(S,+)$; thus if $(S,+)$ is dense in $(\mathbb{R},+)$, we have $S = \mathbb{R}$. For example, when $\phi(x) = \frac{\sin(x)}{\pi x}$, the set $S = \mathbb{R}$.

Before giving the characterization to the case $S = \mathbb{R}$, let us introduce a functional built in [4]. Using the orthogonal projection $P: L^2(\mathbb{R}) \to V$, we defined in [4] a metric functional

$$\sigma(\phi) = \int_0^1 \|P(\phi(x-s))\|^2 ds.$$  

(2.3)

An obvious fact is $S = \mathbb{R} \iff \sigma(\phi) = 1$. But a more interesting and useful fact is that the functional can be simplified as

$$\sigma(\phi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{\phi}(\xi)|^4 \xi d\xi.$$  

(2.4)

Let us quote the proof there.

Proof of (2.4) Writing $P(\phi(x-s)) = \sum_n a_n(s) \phi(x-n)$, and using Parseval’s identity $\langle f, g \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle$, we see

$$\sigma(\phi) = \int_0^1 \sum_n |a_n(s)|^2 ds$$

$$= \int_0^1 \sum_n \|\phi(x-s), \phi(x-n)\|^2 ds$$