Hausdorff Measure of Homogeneous Cantor Set

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Abstract This paper gives the Hausdorff measure of a class of homogeneous Cantor sets.

Keywords Homogeneous cantor set, Hausdorff measure, Convexity

1991MR Subject Classification 28A80

1 Introduction

The determination of the exact Hausdorff measure of a fractal set is a difficult and important problem in fractal geometry. In fact, up to now, only a few results about self-similar sets are known (middle-third Cantor set and its variation, some very special cases of Sierpinski carpets) [1-3]. In this paper we will determine the exact Hausdorff measure of a class of homogeneous Cantor sets by developing the techniques introduced by Mauldin and Williams [4].

Suppose $I = [0, 1]$, let $\{n_k\}_{k \geq 1}$ be a sequence of positive integers, and $\{c_k\}_{k \geq 1}$ be a real number sequence satisfying $n_k \geq 2, 0 < n_k c_k \leq 1 (k \geq 1)$. For any $k \geq 1$, let $D_k = \{(i_1, \ldots, i_k) : 1 \leq i_j \leq n_j, 1 \leq j \leq k\}, D = \bigcup_{k \geq 0} D_k$, where $D_0 = \emptyset$. If $\sigma = (\sigma_1, \ldots, \sigma_k) \in D_k$, $\tau = (\tau_1, \ldots, \tau_m) \in D_m$, let $\sigma \ast \tau = (\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_m)$. Let $F = \{I_\sigma : \sigma \in D\}$ be the collection of the closed sub-intervals of $I$ which satisfy

i) $I_\emptyset = I$;
For any \( k \geq 1 \) and \( \sigma \in D_{k-1} \), \( I_{\sigma+i} \) (\( 1 \leq i \leq n_k \)) are sub-intervals of \( I_\sigma \). \( I_{\sigma+i}, \ldots, I_{\sigma+n_k} \) are arranged from the left to the right, \( I_{\sigma+1} \) and \( I_\sigma \) have the same left endpoint, \( I_{\sigma+n_k} \) and \( I_\sigma \) have the same right endpoint, and the lengths of the gaps between any two consecutive sub-intervals are equal. We denote the length of one of the gaps by \( y_k \).

For any \( k \geq 1 \) and \( \sigma \in D_{k-1} \), \( 1 \leq j \leq n_k \), we have \( \frac{|I_{\sigma+j}|}{|I_\sigma|} = c_k \), where \( |A| \) denotes the diameter of \( A \).

Let \( E_k = \bigcup_{\sigma \in D_k} I_\sigma \), \( E = \bigcap_{k \geq 0} E_k \). We call \( E \) the homogeneous Cantor set [5,6] determined by \( \{n_k\}_{k \geq 1}, \{c_k\}_{k \geq 1} \) and call \( F_k = \{I_\sigma : \sigma \in D_k\} \) the \( k \)-order basic intervals of \( E \).

Feng, Rao and Wu [5] use the net measure method to determine the dimension of \( E \) and give the sufficient and necessary condition for the Hausdorff measure of \( E \) to be positive and finite.

**Theorem A** [5] \ Let \( E \) be the homogeneous Cantor set determined by \( \{n_k\}_{k \geq 1}, \{c_k\}_{k \geq 1} \). Then

1. Let \( s \) be the Hausdorff dimension of \( E \), then
   \[ s = \liminf_{k \to \infty} \frac{\log n_1 \cdots n_k}{\log c_1 \cdots c_k}; \]

2. There exists a constant \( c \), such that
   \[ c \liminf_{k \to \infty} \prod_{j=1}^{k} n_j c_j^s \leq \mathcal{H}^s(E) \leq \liminf_{k \to \infty} \prod_{j=1}^{k} n_j c_j^s, \]

where \( \mathcal{H}^s(E) \) is the \( s \)-dimensional Hausdorff measure of \( E \).

We determine the exact Hausdorff measure of a class of homogeneous Cantor sets. Our main result is

**Theorem 1** \ Let \( E \) be the homogeneous Cantor set determined by \( \{n_k\}_{k \geq 1}, \{c_k\}_{k \geq 1} \). If \( y_{k+1} \leq y_k \) for all \( k \geq 1 \), then

\[ \mathcal{H}^s(E) = \liminf_{k \to \infty} \prod_{j=1}^{k} n_j c_j^s, \]

(1.1)

where \( s \) is the Hausdorff dimension of \( E \).

2 Two Lemmas

Let \( G_k \) be a set consisting of all the possible union of elements in \( F_k \), define

\[ G = \bigcup_{k=0}^{\infty} G_k, \]

(2.1)

and

\[ \mathcal{H}^\alpha_\delta(E) = \liminf_{\delta \to 0} \left\{ \sum |U_i|^\alpha : E \subset \cup U_i, |U_i| < \delta \text{ and } U_i \in G \right\}. \]

(2.2)