On the Conditions of a Center and General Integrals of Quadratic Differential Systems

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Abstract We give the general integral of quadratic differential system with center in two cases under the Chinese classification.

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1 Introduction

As is well known, when a quadratic differential system (QDS, for short) with a weak focus or center is written in the form
\[
\begin{align*}
\dot{x} &= -y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\
\dot{y} &= x + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2,
\end{align*}
\]
the four conditions for O (0, 0) to be a center are [1]:

(i) \( \lambda_4 = \lambda_5 = 0 \), (Hamiltonian) (ii) \( \lambda_3 - \lambda_6 = 0 \), (iii) \( \lambda_2 = \lambda_5 = 0 \),
(iv) \( \lambda_5 = \lambda_4 + 5(\lambda_3 - \lambda_6) = \lambda_3\lambda_6 - 2\lambda_2^2 - \lambda_2^2 = 0 \).

Another version of (1) is [2]:
\[
\begin{align*}
\dot{x} &= -y - bx^2 + Cxy - dy^2, \\
\dot{y} &= x + ax^2 + Axy - ay^2.
\end{align*}
\]
Corresponding to (2), we have now:

\begin{align*}
(i) \quad & A - 2b = C + 2a = 0, \quad \text{(Hamiltonian)} \\
(ii) \quad & b + d = 0, \\
(iii) \quad & C = a = 0, \quad (iv) \quad C + 2a = A + 3b + 5d = a^2 + bd + 2d^2 = 0.
\end{align*}

(4)

Notice that in (1) and (3) the original QDS has already been rotated through a suitable angle such that the coefficients of \(x^2\) and \(y^2\) in the second equation have the same absolute value but are different in sign.

The third version is

\begin{align*}
\dot{x} &= -y - bx^2 - (2c + \beta)xy - dy^2, \\
\dot{y} &= x + ax^2 + (2b + \alpha)xy + cy^2.
\end{align*}

(5)

The conditions for \(O\) to be a center are [3]:

\begin{align*}
\text{(I) } a + c &= b + d = 0, \\
\text{(II) } \alpha(a + c) &= \beta(b + d), \quad \alpha a^3 - (3b + \alpha)\alpha^2 \beta + (3c + \beta)\alpha \beta^2 - d \beta^3 = 0, \\
\text{(III) } \alpha + 5(b + d) &= \beta + 5(a + c) = 2(a^2 + d^2) + ac + bd = 0.
\end{align*}

(6)

Comparing (6) with (4), we see that: (I) is equivalent to (ii); (III) is equivalent to (iv); (i) and (iii) are equivalent to \(a = \beta = 0\) and \(\beta = 0, a = -c = 0\), resp., and they are all contained in (II). But (II) contains more, e.g. when \(\alpha \beta \neq 0\).

The fourth version (Chinese classification) is

\begin{align*}
\dot{x} &= -y + lx^2 + mx y + ny^2, \\
\dot{y} &= x(1 + a_1 x + b_1 y).
\end{align*}

(7)

Comparing (7) with (5), Conditions (6) are now transformed into:

\begin{align*}
\text{(I')} \quad & a_1 = l + n = 0, \\
\text{(II')} \quad & a_1(b_1 + 2l) = m(l + n), a_1(b_1 + 2l)^3 + (b_1 + 2l)^2(b_1 - l)m - (b_1 + 2l + n) m^3 = 0, \\
\text{(III')} \quad & b_1 - 3l - 5n = 5a_1 - m = 2a_1^2 + 2n^2 + ln = 0.
\end{align*}

(8)

Notice that the second equality in (II') is equivalent to

\[m^3[(l + n)2(b_1 + n) - a_1^2(b_1 + 2l + n)] = 0, \quad \text{when } a_1 \neq 0,
\]

(9)

or

\[a_1(b_1 + 2l)^3[(l + n)^2(b_1 + n) - a_1^2(b_1 + 2l + n)] = 0, \quad \text{when } l + n \neq 0.
\]

(10)

So, (8) is equivalent to the following five conditions in Theorem 12.3 of [1]:

\begin{align*}
(i) \quad & a_1 = l + n = 0, \\
(ii) \quad & m(l + n) = a_1(b_1 + 2l), \quad m^3[(l + n)2(b_1 + n) - a_1^2(b_1 + 2l + n)] = 0, \quad a_1 \neq 0, \\
(iii) \quad & m = b_1 + 2l = 0 \quad \text{(Hamiltonian)}, \\
(iv) \quad & m = a_1 = 0, \\
(v) \quad & m = 5a_1 \neq 0, \quad b_1 = 3l + 5n, \quad 2a_1^2 + 2n^2 + ln = 0.
\end{align*}

(11)