A Minimizing Property of Lagrangian Solutions

Shi Qing ZHANG
Department of Mathematics, Chongqing University, Chongqing 400044, P. R. China
E-mail: abc98@cqu.edu.cn

Qing ZHOU
Department of Mathematics, East China Normal University, Shanghai 200062, P. R. China
E-mail: qzhou@math.ecnu.edu.cn

Abstract  In this paper, we prove that the Lagrangian solutions to the three-body problem minimize the action function.

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It is known that the Keplerian orbits minimize the Lagrangian action of the two-body problem (see [1, 2 and 3]), and in this short paper, we will show that the Lagrangian solutions [4] to the three-body problem also minimize the action functional. Note the fact that circular Lagrangian solutions minimize the action functional on the zero mean loop space has already been known (see [5, 6]).

For a given choice of the masses \((m_1, m_2, m_3) \in \mathbb{R}^3_+\), the configuration space of the three-body problem in \(\mathbb{C}\) is given by

\[
F = \left\{ (x_1, x_2, x_3) \in \mathbb{C}^3; \sum_{i=1}^{3} m_i x_i = 0, \text{ and } x_i \neq x_j, i \neq j \right\}.
\]

The loop space \(\mathcal{M}\) we will deal with is the \(W^{1,2}\) completion of the following smooth loop space:

\[
\mathcal{C} = \left\{ (q(t) = (q_1(t), q_2(t), q_3(t)) \in C^\infty(\mathbb{R}/TZ, F); \right.
\]
\[
\left. \text{deg}(q_2 - q_1) \neq 0, \text{deg}(q_3 - q_2) \neq 0, \text{ and } \text{deg}(q_1 - q_3) \neq 0 \right\}.
\]

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and the Lagrangian action is given by

\[ f(q) = \frac{1}{2} \int_0^T \sum_{i=1}^3 m_i |\dot{q}_i|^2 dt + \int_0^T \sum_{i<j} \frac{m_i m_j}{|q_i - q_j|} dt. \]

The solution to the three-body problem is a critical point of the Lagrangian \( f \) on \( \mathcal{M} \). The main result in this paper is the following theorem:

**Theorem 1** The minimal regular solutions to the three-body problem in \( \mathcal{M} \) are precisely the Lagrangian elliptical solutions.

We start with a brief review of Keplerian orbits, Gordon’s result [2] and Lagrangian solutions.

The Keplerian orbits are the regular periodic solutions to the equation

\[ \ddot{x}(t) = -\frac{x(t)}{|x(t)|^3}, \quad x(t) \in C^\infty(\mathbb{R}, \mathbb{C}). \]

It is easy to verify that the energy \( E = \frac{1}{2} |\dot{x}|^2 - |x|^{-1} \) and the angular momentum \( G = x \times \dot{x} \) are constants of the motion (see [1] or [3]). In polar coordinates, the Kepler orbits can be written as

\[ x(t) = r(t) \exp(\sqrt{-1} \alpha(t)), \quad r(t) \text{ and } \alpha(t) \text{ satisfy the equation} \]

\[ r(t) = \frac{G^2}{1 + \sqrt{1 + 2EG^2} \cos(\alpha(t) - \beta)}, \quad G = r^2(t) \dot{\alpha}(t). \]

The curve is an ellipse with eccentricity \( \sqrt{1 + 2EG^2} \), semimajor axis \((-2E)^{-1}\), and the period of the close orbit is \( T = 2\pi(-2E)^{-3/2} \).

The Keplerian orbits are solutions to the two-body problem. Let \( \mathcal{N} \) be the \( W^{1,2} \) completion of the loop space

\[ \{ x(t) \in C^\infty(\mathbb{R}/T\mathbb{Z}, \mathbb{C}), x(t) \neq 0 \text{ and } \deg x \neq 0 \}; \]

and the action functional is defined by

\[ f_1(x) = \int_0^T \left( \frac{|\dot{x}(t)|^2}{2} + \frac{1}{|x(t)|} \right) dt. \]

The critical point of the functional \( f_1 \) in \( \mathcal{N} \) is called the solution to the two-body problem.

Gordon [2] proved the following theorem:

**Theorem 2** The minimal regular solutions to the two-body problem in \( \mathcal{N} \) are precisely the Keplerian orbits, and the minimum of the action functional \( f_1 \) equals \( A = (3\pi)(T/2\pi)^{1/3} \).

For the three-body problem, there are well-known solutions which were discovered by Lagrange [4] in 1772. For three masses \((m_1, m_2, m_3) \in \mathbb{R}_+^3, (m_1 + m_2 + m_3 = 1)\), put the masses at the vertices of an equilateral triangle \( \{t_1, t_2, t_3\} \subset \mathbb{C} \) of side 1 such that \( \sum m_i t_i = 0 \), and pick a Keplerian orbit \( x(t) \); then \( q(t) = x(t)(t_1, t_2, t_3) \) is called a Lagrangian solution. To see that the Lagrangian solution \( q(t) \) is a solution to the three-body problem, we have the following lemma:

**Lemma 3** Lagrangian solutions to the three-body problem minimize the action functional \( f \) in \( \mathcal{M} \).