Notes and Comments

On the uniqueness of convex-ranged probabilities

Massimiliano Amarante

Dept. of Economics, Columbia University, New York
e-mail: ma734@columbia.edu

Received: 10 October 2003 / Accepted: 27 January 2004 – © Springer-Verlag 2004

Mathematics Subject Classification (2000): 28A10, 91B06
Journal of Economic Literature Classification: C60, D81

In Marinacci (2000), the following theorem was proved.

Theorem 1. (Marinacci (2000)) Let $P$ and $Q$ be two finitely additive probabilities on a $\lambda$-system $\Sigma$. Suppose that $P$ is convex-ranged and that $Q$ is countably additive. If there exists an $A^* \in \Sigma$ with $0 < P(A^*) < 1$ such that

$$P(A^*) = P(B) \implies Q(A^*) = Q(B)$$

whenever $B \in \Sigma$, then $P = Q$.

The motivation for studying such question comes from Bayesian decision theory. This is discussed in Marinacci (2000) to which we refer the reader.

In this note, we provide an alternative proof to Marinacci’s theorem by showing that the condition above is equivalent to the condition, that for any $A, B \in \Sigma$, $P(A) > P(B)$ implies that $Q(A) \geq Q(B)$. Lemma 1 below shows that this latter is equivalent to $P = Q$, and Proposition 1 proves the equivalence to Marinacci’s condition. Our condition is probably easier to check in applications. It is in this form, for instance, that the result about the equality of convex-ranged probabilities is repeatedly used in Amarante (2003) for studying the class of unambiguous events in the sense of Epstein and Zhang (2001).
Given the limited scope of this note, we make two assumptions which render the proof entirely elementary. We assume at the outset that both $P$ and $Q$ be nonatomic, countably additive probabilities, and that the class $\Sigma$ of measurable sets is a $\sigma$-algebra. As a matter of fact, that $Q$ is nonatomic not used in the proof of Proposition 1. It could also be dispensed with in the proof of Lemma 1, but at the cost of a lengthier proof. In fact, one could show first that the contrapositive to Lemma 1 does not require that $Q$ be nonatomic, and then proceed as we do below. However, such a proof can be easily obtained by mimicking Marinacci’s argument, and there is no reason for us to do so here. At any rate, the observation is useful. As already noted in Marinacci (2000), the information that both measures are nonatomic is often not available in applications.

Let $P$ and $Q$ be nonatomic, countably additive probabilities on the measurable space $(S, \Sigma)$.

**Lemma 1.** If $P \neq Q$, then there exist $A, B \in \Sigma$ such that

$$P(A) > P(B) \quad \text{and} \quad Q(A) < Q(B).$$

**Proof.** We have $P, Q \ll \mu = \frac{1}{2}P + \frac{1}{2}Q$. Let (Radon-Nikodym) $f_P$ and $f_Q$ be the two densities. Define

$$X = \{s \in S \mid f_P > f_Q\}, \quad Y = \{s \in S \mid f_P < f_Q\}.$$

**Claim.** $X$ and $Y$ are measurable with $\mu(X) > 0$ and $\mu(Y) > 0$.

Measurability is immediate. Let $P \neq Q$. Then there exists $Z \in \Sigma$ such that $P(Z) \neq Q(Z)$. Without loss, assume $P(Z) > Q(Z)$ (and $P(Z^c) < Q(Z^c)$). Now, suppose $\mu(X) = 0$. Then,

$$P(Z) - Q(Z) = \int_Z (f_P - f_Q)d\mu = \int_{Z \cap X} (f_P - f_Q)d\mu + \int_{Z \cap X^c} (f_P - f_Q)d\mu \leq \int_{Z \cap X^c} (f_Q - f_Q)d\mu = 0$$

which implies $P(Z) \leq Q(Z)$, a contradiction. Similarly, for $\mu(Y) > 0$. 