Constrained Diffeomorphic Shape Evolution

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Abstract We design optimal control strategies in spaces of diffeomorphisms and shape spaces in which the Eulerian velocities of the evolving deformations are constrained to belong to a suitably chosen finite-dimensional space, which is also following the motion. This results in a setting that provides a great flexibility in the definition of Riemannian metrics, extending previous approaches in which shape spaces are built as homogeneous spaces under the action of the diffeomorphism group equipped with a right-invariant metric. We provide specific instances of this general setting, and describe in detail the resulting numerical algorithms, with experimental illustrations in the case of plane curves.

Keywords Shape analysis • Optimal control • Groups of diffeomorphisms • Subriemannian geometry

Mathematics Subject Classification (2000) 58D05 • 49N90 • 49Q10 • 68E10

1 Introduction

The estimation of an optimal nonlinear transformation between two deformable objects is an important component of many shape analysis or image comparison meth-
ods. It has become an almost unavoidable processing step in the analysis of anatomical organs from medical images and in computational anatomy [16].

In this setting, it has become increasingly common to estimate such transformations as flows of diffeomorphisms, generated by ordinary differential equations [11, 28, 29]. This approach has led to a family of registration algorithms, called large deformation diffeomorphic metric mapping (LDDMM), in which the correspondence between a template and a target is estimated by minimizing an objective function composed of two terms. The first term is a deformation term that computes what can be interpreted as the total kinetic energy required in the deformation process; the second term is a data term that penalizes the difference between the deformed template and the target. In this construction, the deformation cost at each time $t$ only depends on the Eulerian velocity (the velocity of each particle as a function of its position at time $t$), represented as a time-dependent vector field; it is independent of the deformed object [6, 7, 22, 25, 26, 32]. This method allows one to use classical results in the theory of Lie groups with right-invariant Riemannian metrics [3, 4, 18, 23, 34], yielding Euler–Lagrange equations that are equivalent to the conservation of momentum in time.

In this paper, we describe how deviating from this approach, and allowing the deformation energy to depend on the evolving object, can generate new diffeomorphic flows with specific properties that can be of interest in applications. In particular, we will show that one can conveniently generate such flows by using object-dependent constraints on the space of possible motions at a given time. This will be done by constraining the Eulerian velocity to belong to object-dependent Hilbert subspaces of an underlying Hilbert space of finite-energy velocities. In practice, these spaces will be finitely generated by basis functions (which will be called diffeons) that work as basic finite elements in the construction of diffeomorphisms. This construction will allow us to design metrics that induce more deformation inside a shape than outside it, and vice versa. This will also provide efficient small-dimensional discretizations of the initial numerical problem and open the way to possible multiresolution methods for large-scale problems.

This description will be organized as follows. In Sect. 2, we will introduce the general notation and concepts that lead to flows of diffeomorphisms and the specific construction which is made here to constrain them. This leads in turn to a new class of variational problems, for which the existence of solutions is proved in Sect. 5. After a brief reminder of the kind of solutions that are obtained in the unconstrained case (Sect. 3), in Sect. 4, we describe how constraints can be implemented using diffeons, of which we present some examples. The construction is then embedded in a general optimal control formulation (Sect. 6), which leads to evolution equations and gradient computation via the maximum principle. Explicit choices for the diffeons are presented in Sect. 7, and computational details are provided in Sect. 8. Some quick remarks on the design of data attachment terms are given in Sect. 9, and numerical experiments are presented in Sect. 10, followed by our conclusion.