A generalisation of the Hopf construction and harmonic morphisms into $\mathbb{S}^2$

S. Montaldo · A. Ratto

Abstract In this paper, we construct a new family of harmonic morphisms $\varphi : V^5 \to \mathbb{S}^2$, where $V^5$ is a 5-dimensional open manifold contained in an ellipsoidal hypersurface of $\mathbb{C}^4 = \mathbb{R}^8$. These harmonic morphisms admit a continuous extension to the completion $V^*^5$, which turns out to be an explicit real algebraic variety. We work in the context of a generalization of the Hopf construction and equivariant theory.

Keywords The Hopf construction · Harmonic maps · Harmonic morphisms · Equivariant theory

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1 Introduction

Harmonic maps are critical points of the energy functional

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 dv_g,$$

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where \( \phi : (M, g) \to (N, h) \) is a smooth map between two Riemannian manifolds \( M \) and \( N \). This is a very wide area of research, involving a rich interplay of geometry, analysis and topology. We refer to [2,3,7] for notation and background on harmonic maps.

A geometrically significant sub-family of harmonic maps is represented by the so-called harmonic morphisms, which were first studied by Fuglede and Ishihara in [5,6]; these are maps that preserve the germs of harmonic functions. An exhaustive reference for this topic is the book of Baird and Wood [1], where characterising properties and existence of harmonic morphisms are presented in connection with central themes such as harmonic functions and potential theory; examples of harmonic morphisms include conformal mappings in the plane and holomorphic maps from Kähler manifolds to a Riemann surface.

From an operational point of view, the simplest way to characterise harmonic morphisms is to say that they are just harmonic maps with the additional property that the differential \( d\phi \) is a horizontally weakly conformal map (see [5,6]); this means that, at any point \( x \in M \), either \( d\phi_x \) vanishes or

\[
d\phi_x : (T_x M)_{\mathcal{H}} \to T_{\phi(x)} N
\]

is surjective and conformal. More precisely, in (1.1), \( (T_x M)_{\mathcal{H}} \) denotes the horizontal space \((\ker(d\phi_x))^\perp\), and it is required that there exists a number \( \Lambda_1(x) > 0 \) such that

\[
\Lambda_1(x) g(X, Y) = h(d\phi_x(X), d\phi_x(Y)), \quad \forall X, Y \in (T_x M)_{\mathcal{H}}
\]

The function \( \lambda_1(x) = \sqrt{\Lambda_1(x)} \) is called the dilation (of \( \phi \) at \( x \)). In particular, if \( \phi \) is a non-constant harmonic morphism, then \( m \geq n \), where \( m = \dim(M) \) and \( n = \dim(N) \). Moreover, the set of singular points (i.e., those points where \( d\phi_x = 0 \)) is a closed polar set, i.e., it has zero capacity (see [1] for details).

In this paper, we work in the context of equivariant theory: roughly speaking, it means that we restrict our attention to a class of mappings having enough symmetries to guarantee that harmonicity reduces to the study of a second-order ordinary differential equation. We refer to [4] for notation, background and examples. More specifically, we shall propose a generalisation of the Hopf construction, which gives rise to a new family of harmonic morphisms from a 5-dimensional manifold with singularities onto the Euclidean 2-sphere \( S^2 \).

2 Statement of the main results and related comments

In order to illustrate our framework, we first consider the 3-dimensional ellipsoid

\[
Q^3(a, b) = \left\{ (x, y) \in \mathbb{C} \times \mathbb{C} : \frac{|x|^2}{a^2} + \frac{|y|^2}{b^2} = 1 \right\}, \quad (a, b > 0)
\]

In [4, Chapter X] the authors study in detail, a family of maps

\[
\varphi_{k, \ell} : Q^3(a, b) \to S^2 \subset \mathbb{R}^2 \times \mathbb{R}, \quad k, \ell \in \mathbb{Z}\setminus\{0\}
\]

of the following form (Hopf’s construction)

\[
(a \sin s \ e^{i\theta_1}, b \cos s \ e^{i\theta_2}) \mapsto (\sin \alpha(s) \ e^{i(k\theta_1 + \ell\theta_2)}, \cos \alpha(s))
\]

where the function \( \alpha : (0, \pi/2) \to (0, \pi) \) satisfies the boundary conditions

\[
(i) \quad \lim_{s \to 0^+} \alpha(s) = 0, \quad (ii) \quad \lim_{s \to \pi/2} \alpha(s) = \pi.
\]