On Diophantine equation $3a^2x^4 - By^2 = 1$

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Abstract Bumby proved that the only positive solutions to the quartic Diophantine equation $3x^4 - 2y^2 = 1$ are $(x, y) = (1, 1), (3, 11)$. In this paper, we extend this result and prove that if the class number of the field $Q(\sqrt{1 - 3a^2})$ is not divisible by 2, the equation $3a^2x^4 - By^2 = 1$ has at most two solutions. However, both solutions occur in only one case, $a = 1, b = 2$, as solved by Bumby. The proof utilizes the law of quadratic reciprocity that seems very rare in solving Diophantine equations, and the solution will be also obtained effectively through the proof when it exists.

Keywords Diophantine equation · Quadratic reciprocity law

Mathematics Subject Classification (2000) 11D72

1 Introduction

Let $A, B$ be integers, the Diophantine equation

$$Ax^4 - By^2 = 1$$

has been studied in great detail by many people. For the general case, there is a long-standing conjecture that (1) has at most two integer solutions. Ljuggren [1] proved that (1) has at most two solutions in some cases by the method of algebraic number theory. From then on, there are many other special cases discussed [3–9]. In 1967, Bumby solved the Diophantine equation $3x^4 - 2y^2 = 1$ [2]. His method applies a clever argument involving arithmetic in the quartic number field $Q(\sqrt{-2}, \sqrt{-3})$ and has been regarded to be very difficult to extend to other equations in a long period [5]. The purpose of this paper is to extend Bumby’s method
to the family of the Diophantine equations of type

\[ 3a^2 x^4 - By^2 = 1 \]  \hspace{1cm} (2)

here, we assume the class number of the field \( \mathbb{Q}(\sqrt{1 - 3a^2}) \) is not divisible by 2. (2) includes many interesting special cases such as \( 12x^4 - 11y^2 = 1 \) and \( 300x^4 - 299y^2 = 1 \), which were studied by Bennett and Walsh [5] through elliptic curve with a mass of complicated computations. In [5], the equation \( 12x^4 - 11y^2 = 1 \) was used to find the integer points on the elliptic curve \( dy^2 = x(4x^2 - 3) \) for some \( d \) and this result will be extended in this paper. We also observe an interesting phenomenon about the class number of imaginary quadratic fields which suggests that the result in this paper is best possible. The main result of this paper is:

**Theorem** Let \( a, B \) be positive integers and \( \rho = \sqrt{3a^2} u_0 + \sqrt{B} v_0 > 1 \) be the least positive solution of equation \( 3a^2 u^2 - Bv^2 = 1 \), \( h \) be the class number of the field \( \mathbb{Q}(\sqrt{1 - 3a^2}) \). Suppose that \( h \) is not divisible by 2, then Eq. (2) has at most one solution when \( a \geq 1, B > 2 \), moreover the possible solution is given by \( u_0 = x^2 \), where \( x \) is a rational integer. For \( a = 1, B = 2 \), Eq. (2) has two solutions \((x, y) = (1, 1), (3, 11)\).

**Remark** The latter result of the theorem has been proved by Bumby [2]. In this paper, a slightly different approach is presented.

### 2 Some lemmas and the proof of the theorem

Throughout the paper, we assume \( a \geq 1, B \geq 2 \) are square-free positive integers. At first, we quote an important result from Chen [3] to reduce Eq. (2) to its special case.

**Lemma 1** (Chen Jianhua [3]) Let \( \mu = \sqrt{3a^2} u_0 + \sqrt{B} v_0 > 1 \) be the least solution of the Pell equation

\[ 3a^2 u^2 - Bv^2 = 1 \]  \hspace{1cm} (3)

If (2) is solvable, then \( u_0 = x_0^2 \).

Since all the solutions of (3) are given by

\[
\begin{align*}
  u_k &= u_0 \sqrt{3a^2} + v_0 \sqrt{B} = (u_0 \sqrt{3a^2} + v_0 \sqrt{B})^{2k+1} = \\
  &\quad (\sqrt{3a^2} u_0^2 + \sqrt{3a^2} v_0^2 - 1)^{2k+1} \\
  v_k &= u_0 \sqrt{3a^2} - v_0 \sqrt{B} = (u_0 \sqrt{3a^2} - v_0 \sqrt{B})^{2k+1} = \\
  &\quad (\sqrt{3a^2} u_0^2 - \sqrt{3a^2} v_0^2 - 1)^{2k+1}
\end{align*}
\]

Assume \((x_k, y_k)\) is another solution of (2), then \( u_k = x_k^2 \) and \( u = \frac{u_k}{u_0} \) is a solution of the equation \( 3a^2 u_0^2 u^2 - (3a^2 u_0^2 - 1) v^2 = 1 \). Note that \( u_0 \) is a square, then we reduce (2) to its special case:

\[ 3a^2 x^4 - (3a^2 - 1)y^2 = 1 \]  \hspace{1cm} (4)

In order to solve the equation of the title, we only need to study Eq.(4) instead.

Throughout this paper, we let \( \rho = \sqrt{3a^2} + \sqrt{3a^2 - 1} > 1 \) be the least solution of the Pell equation \( 3a^2 u^2 - (3a^2 - 1) v^2 = 1 \), and \( \theta = \sqrt{-3a^2 + \sqrt{1 - 3a^2}} \). Clearly \( \theta^2 = -\rho^2 \). We have all the solutions of the Pell equation \( 3a^2 u^2 - (3a^2 - 1) v^2 = 1 \) are given by

\[
U_{2k+1} \sqrt{3a^2} + V_{2k+1} \sqrt{3a^2 - 1} = \rho^{2k+1}, \quad k = 0, \pm 1, \pm 2, \ldots.
\]

thus

\[ x^2 = U_{2n-1} = \frac{\rho^{2n} + \rho^{2-2n}}{\rho^2 + 1} = (-1)^n \frac{\theta^{2n} - \theta^{2-2n}}{1 - \theta^2} = (-1)^n \frac{\theta^n - \theta^{1-n}}{1 - \theta} \frac{\theta^n + \theta^{1-n}}{1 + \theta} \]  \hspace{1cm} (5)