Jumping conics on a smooth quadric in \( \mathbb{P}_3 \)

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Received: 16 October 2009 / Accepted: 30 March 2010 / Published online: 13 April 2010
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Abstract We investigate the jumping conics of stable vector bundles \( E \) of rank 2 on a smooth quadric surface \( Q \) with the first Chern class \( c_1 = \mathcal{O}_Q(-1, -1) \) with respect to the ample line bundle \( \mathcal{O}_Q(1, 1) \). We show that the set of jumping conics of \( E \) is a hypersurface of degree \( c_2(E) - 1 \) in \( \mathbb{P}^3_6 \). Using these hypersurfaces, we describe moduli spaces of stable vector bundles of rank 2 on \( Q \) in the cases of lower \( c_2(E) \).

Keywords Jumping conics · Stable bundle · Quadric surface

Mathematics Subject Classification (2000) Primary 14D20; Secondary 14E05

1 Introduction

The moduli space of stable sheaves on surfaces has been studied by many people. Especially, over the projective plane, the moduli space of stable sheaves of rank 2 was studied by Barth [1] and Hulek [10], using the jumping lines and jumping lines of the second kind. In Vitter [18], this idea was generalized to the jumping conics on the projective plane. In this article, we use the concept of jumping conics on the smooth quadric surface, which was introduced, in the case of trivial first Chern class, by Soberon-Chavez in [17]

Let \( Q \) be a smooth quadric in \( \mathbb{P}_3 = \mathbb{P}(V) \), where \( V \) is a 4-dimensional vector space over complex numbers \( \mathbb{C} \), and \( \mathcal{M}(k) \) be the moduli space of stable vector bundles of rank 2 on \( Q \) with the Chern classes \( c_1 = \mathcal{O}_Q(-1, -1) \) and \( c_2 = k \) with respect to the ample line bundle \( H = \mathcal{O}_Q(1, 1) \). \( \mathcal{M}(k) \) forms an open Zariski subset of the projective variety \( \mathcal{M}(k) \) whose points correspond to the semi-stable sheaves on \( Q \) with the same numerical invariants. The Zariski tangent space of \( \mathcal{M}(k) \) at \( E \) is naturally isomorphic to \( H^1(Q, \text{End}(E)) \), and so the dimension of \( \mathcal{M}(k) \) is equal to \( h^1(Q, \text{End}(E)) = 4k - 5 \), since \( E \) is simple.

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Using the Beilinson-type theorem on $Q$ [3], we obtain the following monad for $E \in \mathcal{M}(k)$,
\[0 \to \mathbb{C}^{k-1} \otimes \mathcal{O}_Q(-1, -1) \to \mathbb{C}^k \otimes (\mathcal{O}_Q(0, -1) \oplus \mathcal{O}_Q(-1, 0)) \to \mathbb{C}^{k-1} \otimes \mathcal{O}_Q \to 0,\]
with the cohomology sheaf $E$, where the first injective map derives a map
\[\delta : H^1(E(-1, -1)) \otimes V^* \to H^1(E).\]
As in Barth [2], we similarly define $S(E) \subset \mathbb{P}^3$, the set of jumping conics of $E$, and prove that $S(E)$ is a hypersurface in $\mathbb{P}^3$ of degree $k - 1$ whose equation is given by $\det \delta(z) = 0$, $z \in V^*$, where $\delta(z)$ is a symmetric $(k - 1) \times (k - 1)$-matrix. We give a criterion for $H \in \mathbb{P}^*$ to be a singular point of $S(E)$ and calculate the exact number of singular points of $S(E)$ when $E$ is a Hulsbergen bundle, i.e. $E$ admits the following exact sequence,
\[0 \to \mathcal{O}_Q \to E(1, 1) \to I_Z(1, 1) \to 0,\]
where $Z$ is a 0-cycle on $Q$ with length $k$ whose support is in general position.

In Sect. 3, we describe the above results in the cases $c_2 \leq 3$ by investigating the map
\[S : \mathcal{M}(k) \to \mid \mathcal{O}_{\mathbb{P}^3}(k - 1),\]
sending $E$ to $S(E)$. When $c_2 = 2$, $S(E)$ is a hypersurface in $\mathbb{P}^3$, and $\mathcal{M}(2)$ is isomorphic to $\mathbb{P}_3 \setminus Q$ via $S$, which was already shown in Huh [9]. In the case of $c_2 = 3$, we investigate the surjective map from $\mathcal{M}(3)$ to $\mathbb{P}^3$, sending $E$ to the vertex point of the quadric cone $S(E) \subset \mathbb{P}^3$ to give an explicit description of $\mathcal{M}(3)$. In fact, the generic fibre of this map over $H \in \mathbb{P}^*$ is isomorphic to the set of smooth conics that are Poncelet related to the smooth conic $H \cap Q$.

As a result, we can observe that $S$ is generically one to one from $\mathcal{M}(3)$ to its image. In other words, when $c_2 = 2, 3$, the set of jumping conics, $S(E)$, uniquely determines $E$ in general.

### 2 The Beilinson theorem and jumping conics

#### 2.1 The Beilinson theorem

Let $V_1$ and $V_2$ be two 2-dimensional vector spaces with the coordinate $[x_{1j}]$ and $[x_{2j}]$, respectively. Let $Q$ be a smooth quadric isomorphic to $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$, and then it is embedded into $\mathbb{P}_3 \cong \mathbb{P}(V)$ by the Segre map, where $V = V_1 \otimes V_2$. Let us denote $f^* \mathcal{O}_{\mathbb{P}_1}(a) \otimes g^* \mathcal{O}_{\mathbb{P}_1}(b)$ by $\mathcal{O}_Q(a, b)$ and $E \otimes \mathcal{O}_Q(a, b)$ by $E(a, b)$ for coherent sheaves $E$ on $Q$, where $f$ and $g$ are the projections from $Q$ to each factor. Then, the canonical line bundle $K_Q$ of $Q$ is $\mathcal{O}_Q(-2, -2)$.

**Definition 2.1** For a fixed ample line bundle $H$ on $Q$, a torsion-free sheaf $E$ of rank $r$ on $Q$ is called stable (resp. semi-stable) with respect to $H$ if
\[\frac{\chi(F \otimes \mathcal{O}_Q(mH))}{r'} < (\text{resp. } \leq) \frac{\chi(E \otimes \mathcal{O}_Q(mH))}{r},\]
for all non-zero subsheaves $F \subset E$ of rank $r'$.

Let $\overline{\mathcal{M}}(k)$ be the moduli space of semi-stable sheaves of rank 2 on $Q$ with the Chern classes $c_1 = \mathcal{O}_Q(-1, -1)$ and $c_2 = k$ with respect to the ample line bundle $H = \mathcal{O}_Q(1, 1)$. The existence and the projectivity of $\overline{\mathcal{M}}(k)$ is known in Gieseker [7], and it has an open Zariski subset $\mathcal{M}(k)$ that consists of the stable vector bundles with the given numeric invariants. By the Bogomolov theorem, $\mathcal{M}(k)$ is empty if $4k < c_1^2 = 2$, and in particular, we can consider