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Radial solutions to the wave equation

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Abstract. We study some boundedness properties of radial solutions to the Cauchy problem associated to the wave equation \((\partial_t^2 - \Delta_x)u(t, x) = 0\) and meanwhile we give a new proof of the solution formula.

Sunto. Studiamo delle proprietà di limitatezza per soluzioni radiali del problema di Cauchy associato all’equazione delle onde \((\partial_t^2 - \Delta_x)u(t, x) = 0\) e nel frattempo diamo anche una nuova dimostrazione della formula risolutiva per tale equazione.

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In this paper we want to prove some norm estimates for radial solutions to the wave equation and, in order to do this, we start by recalling the explicit form of these solutions.

Let us consider the Cauchy problem for the wave equation in \((t, x) \in \mathbb{R} \times \mathbb{R}^D\),

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} u (t, x) &= \sum_{j=1}^{D} \frac{\partial^2}{\partial x_j^2} u (t, x), \\
u (0, x) &= \Phi (x), \quad \frac{\partial}{\partial t} u (0, x) = \Psi (x).
\end{align*}
\]

The d’Alembert formula,

\[
u(t, x) = \frac{\Phi(x - t) + \Phi(x + t)}{2} + \frac{1}{2} \int_{x - t}^{x + t} \Psi(s) ds,
\]

gives the solution when the space dimension is one, while the Poisson formula,

\[
u(t, x) = \frac{\partial}{\partial t} \left\{ \frac{t}{4\pi} \int_{|y|=1} \Phi(x - ty) d\sigma(y) \right\} + \left\{ \frac{t}{4\pi} \int_{|y|=1} \Psi(x - ty) d\sigma(y) \right\}.
\]

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gives the solution in dimension three. The form of solutions in an arbitrary dimension is due to Tedone (see [10]). When the dimension is odd, $D = 2N + 1$, then

$$u(t, x) = b(N) \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial r} \right)^{N-1} \left\{ r^{2N-1} \int_{|y|=1} \Phi(x - ty) d\sigma(y) \right\}$$

$$+ b(N) \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{N-1} \left\{ r^{2N-1} \int_{|y|=1} \Psi(x - ty) d\sigma(y) \right\},$$

where $b(N) = 2^{-1} (1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2N - 1))^{-1} \pi^{-N/2} \Gamma(N + 1/2)$ and $d\sigma(y)$ is the surface measure on $\{|y|=1\}$. When the dimension is even, $D = 2N$, by the method of descent one has

$$u(t, x) = 2b(N) \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial r} \right)^{N-1} \left\{ r^{2N-1} \int_{|y|<1} \Phi(x - ty) \frac{1}{\sqrt{1 - |y|^2}} dy \right\}$$

$$+ 2b(N) \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{N-1} \left\{ r^{2N-1} \int_{|y|<1} \Psi(x - ty) \frac{1}{\sqrt{1 - |y|^2}} dy \right\}.$$

These formulas require approximately $D/2$ derivatives of the initial conditions, hence they are not immediately applicable to non-smooth data. However, it is known that when these data are radial then there exists a solution formula with no derivatives involved. When the initial data are radial, writing $|x| = r$, $\Phi(x) = \phi(r)$, $\Psi(x) = \psi(r)$, with $\phi(r)$ and $\psi(r)$ even functions on $R$, and expressing the Laplacian in polar coordinates, one is led to the Cauchy problem

$$\begin{align*}
\frac{\partial^2}{\partial t^2} u(t, r) &= \frac{\partial^2}{\partial r^2} u(t, r) + \frac{D - 1}{r} \frac{\partial}{\partial r} u(t, r), \\
u(0, r) &= \phi(r), \quad \frac{\partial}{\partial t} u(0, r) = \psi(r).
\end{align*}$$

Assume now $D = 3$. Then one easily checks that $r \cdot u(t, r)$ satisfies the one-dimensional wave equation $r^2 u(t, r) = \frac{\partial^2}{\partial r^2} (ru(t, r))$ with $ru(0, r) = r\phi(r)$ and $\frac{\partial}{\partial t} (ru(0, r)) = r\psi(r)$. Hence, by d’Alembert’s formula,

$$u(t, r) = \frac{(r - t)\phi(r - t) + (r + t)\phi(r + t)}{2r} + \frac{1}{2r} \int_{r-t}^{r+t} s\psi(s) ds.$$

Observe that since $s\phi(s)$ is an odd function, the above integration can be reduced to the interval $|r - t| < s < r + t$. This formula does not contain derivatives of the initial data, at least if $r \neq 0$, but taking the limit as $r \to 0+$ a derivative appears, $u(t, 0) = t \frac{\partial}{\partial t} \phi(t) + \phi(t) + t\psi(t)$.

For an arbitrary dimension the formula of solutions is more complicated. It involves terms of the form $r^{(1-D)/2} |r \pm t|^{(D-1)/2} \phi(r \pm t)$ and integrals of the