KHovanov homology is an unknot-detector
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ABSTRACT

We prove that a knot is the unknot if and only if its reduced Khovanov cohomology has rank 1. The proof has two steps. We show first that there is a spectral sequence beginning with the reduced Khovanov cohomology and abutting to a knot homology defined using singular instantons. We then show that the latter homology is isomorphic to the instanton Floer homology of the sutured knot complement: an invariant that is already known to detect the unknot.

1. Introduction

1.1. Statement of results

This paper explores a relationship between the Khovanov cohomology of a knot, as defined in [16], and various homology theories defined using Yang-Mills instantons, of which the archetype is Floer’s instanton homology of 3-manifolds [8]. A consequence of this relationship is a proof that Khovanov cohomology detects the unknot. (For related results, see [10–12].)

Theorem 1.1. — A knot in $S^3$ is the unknot if and only if its reduced Khovanov cohomology is $\mathbb{Z}$.

In [21], the authors construct a Floer homology for knots and links in 3-manifolds using moduli spaces of connections with singularities in codimension 2. (The locus of the singularity is essentially the link $K$, or $\mathbb{R} \times K$ in a cylindrical 4-manifold.) Several variations of this construction are already considered in [21], but we will introduce here one more variation, which we call $I^\mathbb{R}(K)$. Our invariant $I^\mathbb{R}(K)$ is an invariant for unoriented links $K \subset S^3$ with a marked point $x \in K$ and a preferred normal vector $v$ to $K$ at $x$. The purpose of the normal vector is in making the invariant functorial for link cobordisms: if $S \subset [0,1] \times S^3$ is a link cobordism from $K_1$ to $K_0$, not necessarily orientable, but equipped with a path $\gamma$ joining the respective basepoints and a section $v$ of the normal bundle to $S$ along $\gamma$, then there is an induced map,

$$I^\mathbb{R}(K_1) \to I^\mathbb{R}(K_0)$$

that is well-defined up to an overall sign and satisfies a composition law. (We will discuss what is needed to resolve the sign ambiguity in Section 4.4.) The definition is set up so that $I^\mathbb{R}(K) = \mathbb{Z}$ when $K$ is the unknot. We will refer to this homology theory as the

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reduced singular instanton knot homology of $K$. (There is also an unreduced version which we call $\Gamma(K)$ and which can be obtained by applying $\Gamma$ to the union of $K$ with an extra unknotted, unlinked component.) The definitions can be extended by replacing $S^3$ with an arbitrary closed, oriented 3-manifold $Y$. The invariants are then functorial for suitable cobordisms of pairs.

Our main result concerning $\Gamma(K)$ is that it is related to reduced Khovanov cohomology by a spectral sequence. The model for this result is a closely-related theorem due to Ozsváth and Szabó [29] concerning the Heegaard Floer homology, with $\mathbb{Z}/2$ coefficients, of a branched double cover of $S^3$. There is a counterpart to the result of [29] in the context of Seiberg-Witten gauge theory, due to Bloom [3].

**Proposition 1.2.** — With $\mathbb{Z}$ coefficients, there is a spectral sequence whose $E_2$ term is the reduced Khovanov cohomology, $\text{Khr}(\bar{K})$, of the mirror image knot $\bar{K}$, and which abuts to the reduced singular instanton homology $\Gamma(K)$.

As an immediate corollary, we have:

**Corollary 1.3.** — The rank of the reduced Khovanov cohomology $\text{Khr}(K)$ is at least as large as the rank of $\Gamma(K)$.

To prove Theorem 1.1, it will therefore suffice to show that $\Gamma(K)$ has rank bigger than 1 for non-trivial knots. This will be done by relating $\Gamma(K)$ to a knot homology that was constructed from a different point of view (without singular instantons) by Floer in [9]. Floer’s knot homology was revisited by the authors in [23], where it appears as an invariant $\text{KHI}(K)$ of knots in $S^3$. (There is a slight difference between $\text{KHI}(K)$ and Floer’s original version, in that the latter leads to a group with twice the rank.) It is defined using $\text{SU}(2)$ gauge theory on a closed 3-manifold obtained from the knot complement. The construction of $\text{KHI}(K)$ in [23] was motivated by Juhász’s work on sutured manifolds in the setting of Heegaard Floer theory [13, 14]: in the context of sutured manifolds, $\text{KHI}(K)$ can be defined as the instanton Floer homology of the sutured 3-manifold obtained from the knot complement by placing two meridional sutures on the torus boundary. It is defined in [23] using complex coefficients for convenience, but one can just as well use $\mathbb{Q}$ or $\mathbb{Z}[1/2]$. The authors establish in [23] that $\text{KHI}(K)$ has rank larger than 1 if $K$ is non-trivial. The proof of Theorem 1.1 is therefore completed by the following proposition (whose proof turns out to be a rather straightforward application of the excision property of instanton Floer homology).

**Proposition 1.4.** — With $\mathbb{Q}$ coefficients, there is an isomorphism between the singular instanton homology $\Gamma(K; \mathbb{Q})$ and the sutured instanton homology of the knot complement, $\text{KHI}(K; \mathbb{Q})$.

**Remark.** — We will see later in this paper that one can define a version of $\text{KHI}(K)$ over $\mathbb{Z}$. The above proposition can then be reformulated as an isomorphism over $\mathbb{Z}$ between $\Gamma(K)$ and $\text{KHI}(K)$.