1. Introduction

The shortest loop traced out by a billiard ball in an acute triangle is the pedal subtriangle, connecting the feet of the altitudes.

In this paper we prove a similar result for loops in the fundamental polyhedron of a Coxeter group $W$, and use it to study the spectral radius $\lambda(w)$, $w \in W$ for the geometric action of $W$. In particular we prove:

**Theorem 1.1.** — Let $(W, S)$ be a Coxeter system and let $w \in W$. Then either $\lambda(w) = 1$ or $\lambda(w) \geq \lambda_{Lehmer} \approx 1.1762808$.

Here $\lambda_{Lehmer}$ denotes *Lehmer’s number*, a root of the polynomial

\[
1 + x - x^3 - x^4 - x^5 - x^6 - x^7 + x^9 + x^{10}
\]

and the smallest known Salem number.

**Billiards.** — Recall that a *Coxeter system* $(W, S)$ is a group $W$ with a finite generating set $S = \{s_1, \ldots, s_n\}$, subject only to the relations $(s_is_j)^{m_{ij}} = 1$, where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. The permuted products

$$s_{\sigma_1}s_{\sigma_2} \cdots s_{\sigma_n} \in W, \quad \sigma \in S_n,$$

are the *Coxeter elements* of $(W, S)$. We say $w \in W$ is *essential* if it is not conjugate into any subgroup $W_I \subseteq W$ generated by a proper subset $I \subset S$.
The Coxeter group $W$ acts naturally by reflections on $V \cong \mathbb{R}^8$, preserving an inner product $B(v, v')$. Let $\lambda(w)$ denote the spectral radius of $w|V$. When $\lambda(w) > 1$, it is also an eigenvalue of $w$. We will show (§4):

**Theorem 1.2.** — Let $(W, S)$ be a Coxeter system and let $w \in W$ be essential. Then we have $\lambda(w) \geq \inf_{s_\sigma} \lambda(s_{\sigma_1}s_{\sigma_2} \cdots s_{\sigma_n})$.

Here is the relation to billiards. In the case of a hyperbolic Coxeter system (when $(V, B)$ has signature $(p, 1)$), the orbifold $Y = \mathbb{H}^p/W$ is a convex polyhedron bounded by mirrors meeting in acute angles. Closed geodesics on $Y$ can be visualized as loops traced out by billiards in this polyhedron. The hyperbolic length of the geodesic in the homotopy class of $w \in \pi_1(Y) = W$ is given by $\log \lambda(w)$. Thus the theorem states that the essential billiard loops in $Y$ are no shorter than the shortest Coxeter element.

As a special (elementary) case, the shortest billiard loop in the $(p, q, r)$-triangle in $\mathbb{H}^2$ is the pedal subtriangle representing $w = s_1s_2s_3$; see Figure 1.

**The Hilbert metric on the Tits cone.** — To prove Theorem 1.2 for higher-rank Coxeter groups (signature $(p, q)$, $q \geq 2$), we need a generalization of hyperbolic space. A natural geometry in this case is provided by the Hilbert metric on the Tits cone.

The Hilbert distance on the interior of a convex cone $K$ is given in terms of the cross-ratio by $d_K(x, y) = (1/2) \inf \log[a, x, y, b]$, where the infimum is over all segments $[a, b]$ in $K$ containing $[x, y]$; it is a metric when $K$ contains no line. We will show (§2)