Optimal Control Problem Governed by Semilinear Parabolic Equation and its Algorithm

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Abstract In this paper, an optimal control problem governed by semilinear parabolic equation which involves the control variable acting on forcing term and coefficients appearing in the higher order derivative terms is formulated and analyzed. The strong variation method, due originally to Mayne et al to solve the optimal control problem of a lumped parameter system, is extended to solve an optimal control problem governed by semilinear parabolic equation, a necessary condition is obtained, the strong variation algorithm for this optimal control problem is presented, and the corresponding convergence result of the algorithm is verified.

Keywords Distributed parameter system, strong variation method, optimal control, adjoint system, semilinear parabolic equations

2000 MR Subject Classification 49k15, 49k20

1 Introduction

In the last decade, optimal control of nonlinear parabolic distributed parameter system is a growing area of research⁴–⁵⁹. By using different approaches (weak solutions, viscosity solutions, · · ·) and methods (penalization, linearization, relaxation, · · ·), different kinds of optimality conditions have been obtained. The existence and uniqueness under different types and conditions were also studied. He and Yong investigated the optimal control problems of semilinear and quasilinear parabolic with state constraint, and the corresponding Pontryagin maximum principle was given in [⁷]; Gao Hang discussed the first order necessary condition of a class of optimal control problem for the semilinear parabolic type with the control variable acting on coefficients appearing in the higher order derivative terms of the differential equation [⁶]. However, the research of the algorithms of the optimal control problem of distributed parameter system far lags behind that of the associated optimality conditions. As far as we know, the first paper in strong variation algorithm for the optimal control was written by Maybe et al.[¹⁰], in which the strong variation algorithm involving the optimal control of lumped parameter system was studied. In [²], Ahmed et al investigated the optimal control problem of a linear parabolic equation with first and second boundary conditions, constructed an algorithm to deal with this kind of optimal control problem and finally proved that a sequence $u^k$ of controls can be constructed so that $J(u^{k+1}) \leq J(u^k)$ for all $k(k = 1, 2, \cdots)$, where $J$ is the objective functional[¹²]. However, no convergence results on $u^k$ were given; The above mentioned algorithm was extended by Wong to the optimal control problem of a class of linear parabolic systems involving...
In this paper, we investigate an optimal control problem governed by a semilinear parabolic equation which involves the control variable acting on forcing term and coefficients appearing in the higher order derivative terms. We prove that, for every non-extreme admissible control function, another admissible control function can be constructed so that the corresponding value of the objective functional is less than that of the former; According to the above result, we obtain a necessary condition for this kind of optimal control problem; At last, we construct the corresponding algorithm and verify the convergence of the algorithm.

The rest of the paper is organized as follows: in Section 2 we study the state equation and formulate the control problem under consideration; in Section 3, we prove that, for every non-extreme admissible control function, another admissible control function can be constructed so that the value of the corresponding objective functional is less than that of the former and deduce the optimality necessary conditions of the optimal control problem. Finally, in Section 4, we present a concrete algorithm for the optimal control problem and the convergence of the algorithm is verified.

2 Formulation of the Problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded, closed convex domain with sufficiently smooth boundary $\partial \Omega$, $W \subset \mathbb{R}^m$ be a bounded closed set, $T > 0$ be a fixed time. Let $Q = \Omega \times (0, T)$, $\Sigma = \partial \Omega \times (0, T)$. Now we consider a class of optimal control problem governed by the following semilinear parabolic system.

\begin{align*}
L(u)y(x, t) &= f(x, t, y(x, t), u(x, t)), \quad (x, t) \in Q, \\
y(x, t) &= 0, \quad (x, t) \in \Sigma, \\
y(x, 0) &= y_0(x), \quad x \in \Omega,
\end{align*}

where

\[ L(u)y(x, t) = \frac{\partial}{\partial t}y(x, t) - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x, t, u(x, t)) \frac{\partial}{\partial x_j} y(x, t) \right), \]

$y(x, t), u(x, t)$ are the state and the control function respectively. Let $u : Q \to W$ be a measurable function, $W$ denote the set including all the above measurable functions, namely

\[ W = \{ u | u : Q \to W \text{ are measurable functions} \}. \]

Consider the following objective functional

\[ J(u) = \int_{Q} g(x, t, y(x, t, u), \nabla y(x, t, u), u(x, t)) \, dx \, dt \]

where $y(x, t, u)$ is the solution to Problem (1) with $u \in W$.

Throughout this paper, we adopt the following assumptions.

**A1:** $a_{ij} : Q \times W \to \mathbb{R}^1$ satisfies $a_{ij} (\cdot, u) \in C(\overline{Q}), a_{ij} (x, t, \cdot)$ is a Lipschitz continuous function with Lipschitz coefficient $\mathcal{T}$, and there exists constants $C_1, C_2 > 0$ such that, for arbitrary $(x, t, u) \in Q \times W, \eta \in \mathbb{R}^n$,

\[ C_1 \| \eta \|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x, t, u) \eta_i \eta_j \leq C_2 \| \eta \|^2. \]