CLS asymptotic variance for a particular relevant bilinear time series model

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Abstract. The main purpose of this paper is to gain inference about the parameter estimated via the CLS method for a particular bilinear model. We propose a new CLS estimator which is strongly consistent and the CLT and LIL hold with milder conditions on the moments. Furthermore, we derive a closed form expression for the asymptotic variance of this CLS estimator.

Key words: Bilinear model, CLS estimators, Asymptotic estimator distribution

1. Introduction

It is widely accepted (see Takens, 1981) that a time series $X_t$ can be decomposed as $X_t = S_t + e_t$, where $S_t$ is the signal and $e_t$ the error term. In general it is possible to write

$$X_t = f(X_{t-\tau}, X_{t-2\tau}, \ldots, X_{t-k\tau}, Y_t, Y_{t-\delta}, \ldots, Y_{t-h\delta}; \beta) + e_t$$

(1)

where $Y_{t-\delta}$, with $\delta \geq 0$, are lagged exogenous variables, $f(.)$ is a possibly non linear function and $\beta$ is a vector of parameters.

Model (1) is able to reproduce the non linear dynamics in the following three alternative ways:

The non linearity depends on the shape of $f(.)$. In this case the residual component $e_t$ can be supposed Normal with $E(e_t) = 0$, $Var(e_t) = \sigma^2$ and $Cov(e_t, e_s) = 0$ for $t \neq s$. This kind of non linearity is typical for Threshold AutoRegressive (TAR) models;

$f(.)$ is a linear function but $e_t$ has a non linear structure such as the bilinear one. This is the case of ARMA models with non normal residuals;

$f(.)$ and $e_t$ are both non linear. The residuals of these models can be easily checked for their non linearity and their structure cannot be explained by $f(.)$. The Threshold AR, with bilinear residuals (TAR-BL), is a typical example for this class.
The last case is very interesting because the non linear structure in the residuals can also be used to model an asymmetric conditional heteroskedasticity (Tong, 1990).

It is known that the bilinear models (Tong, 1990) are characterised by a very complex probability structure. Besides, the estimates are not stable to generate forecasts and the estimators have not an explicit form. Taking into account a particular bilinear model this can be thought as a generalisation of the Normal residual case with heavier tails. In fact, a flexible model for the error term in dynamic systems is given by the following simple bilinear model:

\[ X_t = b\varepsilon_{t-1}X_{t-2} + \varepsilon_t \]  

(2)

where \( \varepsilon_t \sim N(0, \sigma^2) \forall t, \text{Cov}(\varepsilon_t, \varepsilon_s) = 0 \) for \( t \neq s \), with \( b \) and \( \sigma^2 \) being unknown parameters to be estimated.

The estimation of (1) is performed in two steps:

the estimation of the signal;
the fitting of model (2) to the residuals.

Let \( b_t = b\varepsilon_{t-1} \) be a stochastic parameter, then model (2) is an AR(1) with a random coefficient given by \( b_t \). Grahn (1995) showed, for a more general bilinear model, the strong convergence and the normality of Conditional Least Squares (CLS) estimators. Nevertheless there is no expression to estimate the asymptotic variance of these CLS estimators even if we consider the simple case (2).

For the stochastic process \( X_t \) in (2) it is

\[ E(X_t) = 0 \]

\[ Var(X_t) = \mu_X^2 \]

\[ Cov(X_t, X_{t+k}) = 0 \quad \forall t \forall k \neq 0 \]

and \( X_t \) may be confused with a White Noise process.

This paper illustrates a proposal for the definition of a closed form expression for the asymptotic variance of the CLS estimator of the parameter \( b \) in (2), which is also useful to work out some properties of the residuals generated by different models.

In Sect. 2 we suggest an estimator \( \hat{b}_r \) for the parameter \( b \) which owns the same properties of the “classical” Grahn estimator but imposes milder moment conditions. Besides, an expression for the asymptotic variance of the estimator \( \hat{b}_r \) is derived. In Sect. 3 we show the results of a simulation study performed in order to evaluate the asymptotic variance against the simulated one. Thus an estimator of this asymptotic variance is considered and its properties are investigated for different values of the sample and of the parameter \( b \). Some remarks about the results are given in the final Sect. 4.

2. Asymptotic variance for a CLS estimator of \( b \)

After some preliminaries about the process defined in (2), we introduce a new CLS estimator of the parameter \( b \). Hence we derive a closed form expression for the