A hybrid-stress element based on Hamilton principle

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Abstract A novel hybrid-stress finite element method is proposed for constructing simple 4-node quadrilateral plane elements, and the new element is denoted as HH4-3β here. Firstly, the theoretical basis of the traditional hybrid-stress elements, i.e., the Hellinger–Reissner variational principle, is replaced by the Hamilton variational principle, in which the number of the stress variables is reduced from 3 to 2. Secondly, three stress parameters and corresponding trial functions are introduced into the system equations. Thirdly, the displacement fields of the conventional bilinear isoparametric element are employed in the new models. Finally, from the stationary condition, the stress parameters can be expressed in terms of the displacement parameters, and thus the new element stiffness matrices can be obtained. Since the required number of stress variables in the Hamilton variational principle is less than that in the Hellinger–Reissner variational principle, and no additional incompatible displacement modes are considered, the new hybrid-stress element is simpler than the traditional ones. Furthermore, in order to improve the accuracy of the stress solutions, two enhanced post-processing schemes are also proposed for element HH4-3β. Numerical examples show that the proposed model exhibits great improvements in both displacement and stress solutions, implying that the proposed technique is an effective way for developing simple finite element models with high performance.

Keywords Finite element · Hybrid-stress element · Hamilton variational principle · Post-processing schemes

1 Introduction

Over the past 50 years, the finite element method has been developed into a powerful and widely-used numerical tool in modern designs and simulations for engineering problems [1,2]. In many practical applications, the lower-order element models are often preferred because of their high computation efficiency. However, these simple models often suffer from many numerical problems, such as various locking phenomena brought about by full integration scheme, and the hourglass problem arising from reduced integration scheme.

Many efforts have been made for improving the performance of the simple bilinear 4-node quadrilateral isoparametric element, such as the hybrid method proposed by Pian
the field variables are composed of two displacements \((u, v)\) and three stresses \((\sigma_x, \sigma_y, \tau_{xy})\). Then, the energy functional \(\Pi_{HR}\) of the Hellinger–Reissner variational principle for plane stress problem can be written as

\[
\Pi_{HR} = h \int_A \left\{ \left[ \sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \tau_{xy} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \\
- \frac{1}{2E} \left[ \sigma_x^2 + \sigma_y^2 - 2\mu \sigma_x \sigma_y + 2(1 + \mu) \tau_{xy}^2 \right] \\
- \tilde{F}_x u - \tilde{F}_y v \right\} \, dA + b,
\]

where \(h\) is the thickness; \(\tilde{F}_x\) and \(\tilde{F}_y\) are body forces; \(E\) and \(\mu\) are elasticity modulus and Poisson’s ratio, respectively. (For a plane strain problem, the \(E\) and \(\mu\) in Eq. (1) should be replaced by \(E/(1-\mu^2)\) and \(\mu/(1-\mu)\), respectively). The last term \(b\) denotes the boundary terms

\[
b = -h \int_{S_b} \left( \tilde{T}_x u + \tilde{T}_y v \right) \, ds \\
- h \int_{S_b} \left[ T_x (u - \bar{u}) + T_y (v - \bar{v}) \right] \, ds,
\]

in which \(\bar{T}_x\) and \(\bar{T}_y\) are the specified boundary tractions; \(\bar{u}\) and \(\bar{v}\) are the specified displacements; \(T_x\) and \(T_y\) are boundary tractions along the displacement boundaries, and can be expressed as

\[
T_x = l \sigma_x + m \tau_{xy}, \quad T_y = m \sigma_y + l \tau_{xy},
\]

where \(l\) and \(m\) are the direction cosines of the displacement boundaries.

From the stationary condition \(\delta \Pi_{HR} = 0\), the following stress–displacement relations can be obtained

\[
\frac{\partial u}{\partial x} = \frac{1}{E} \left( \sigma_x - \mu \sigma_y \right), \\
\frac{\partial v}{\partial y} = \frac{1}{E} \left( \sigma_y - \mu \sigma_x \right), \\
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{2(1 + \mu)}{E} \tau_{xy}.
\]

In the Hamilton variational principle and solution system, which are usually adopted for dynamic problems, have been utilized by some researchers to establish new analytical solutions for elastic problems [22–26]. In this paper, a plane hybrid-stress element method based on the Hamilton variational principle is proposed for constructing simple 4-node quadrilateral plane elements. Since the theoretical basis of the hybrid-stress elements, i.e., the Hellinger–Reissner variational principle, is replaced by the Hamilton variational principle, the number of the stress variables can be reduced from 3 to 2, and thus the new hybrid-stress elements are simpler than the traditional ones. Furthermore, several enhanced post-processing schemes [27–31] are employed for improving the stress accuracy of the new element, and the performance of the proposed elements are finally validated by selected numerical examples.

### 2 Hamilton functional for 2D isotropic elasticity

In the Hellinger–Reissner solution system for 2D elasticity, the field variables are composed of two displacements \((u, v)\) and three stresses \((\sigma_x, \sigma_y, \tau_{xy})\). Then, the energy functional \(\Pi_{HR}\) of the Hellinger–Reissner variational principle for plane stress problem can be written as

\[
\Pi_{HR} = h \int_A \left\{ \frac{1}{2} \left[ \sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \tau_{xy} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \\
- \frac{1}{2E} \left[ \sigma_x^2 + \sigma_y^2 - 2\mu \sigma_x \sigma_y + 2(1 + \mu) \tau_{xy}^2 \right] \\
- \tilde{F}_x u - \tilde{F}_y v \right\} \, dA + b,
\]