Soliton Equations and Simple Combinatorics

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Received: 18 December 2007 / Accepted: 10 January 2008 / Published online: 23 January 2008
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Abstract A systematic, elementary and pedagogical approach to a class of soliton equations, and to their spectral formulation, is presented. This approach, based on the use of exponential polynomials, follows naturally from a comparison of some simple results for two representatives of the class: the KdV- and the Boussinesq-equation.

Keywords Soliton · NLPDE · Integrable systems · Bäcklund transformation · Lax pairs

AMS Subject Classification 35Q51 · 37K40 · 14H70

1 Introduction

Soliton theory originated in the discovery that a classical nonlinear wave equation, introduced by Korteweg and de Vries (KdV) for the description of long wave propagation on shallow water [24], is intimately connected with a linear Schrödinger-like eigenvalue problem [12].

In particular, it was realized that the KdV-equation:

\[ \text{KdV}(u) \equiv u_t - u_{3x} - 6uu_x = 0, \quad u_{px} = \frac{\partial^p u}{\partial x^p}, \]  

(1)

coincides with the compatibility condition on the following pair of linear differential equations [26]:

\[ L_2(u) \phi = \lambda \phi, \quad L_2(u) = \partial_x^2 + u, \]  

(2)

\[ \phi_t = B_3(u) \phi, \quad B_3(u) = 4\partial_x^3 + 6u\partial_x + 3u_x, \]  

(3)
and that it may be reformulated as an operator equation, involving the commutator of the corresponding “Lax pair” \((L_2, B_3)\):

\[
[\partial_t - B_3, L_2] = 0.
\] (4)

The disclosure of this remarkable connection was triggered by the observation, by Miura [29], that KdV admits solutions \(u\), which are produced by solutions \(\sigma\) to a “modified” version of KdV:

\[
m\text{KdV}(\sigma) \equiv \sigma_t - \sigma_{3x} + 6\sigma^2\sigma_x = 0,
\] (5)

by means of the logarithmically linearizable transformation:

\[
u = -\sigma_x - \sigma^2.
\] (6)

The Lax formulation \((2, 3)\) of \((1)\) led to a spectral interpretation of its sech squared soliton solutions (\(k\) and \(\eta\) are arbitrary parameters):

\[
u_{\text{sol}} = \frac{k^2}{2} \text{sech}^2 \left( \frac{\theta}{2} \right), \quad \theta = kx + k^3t + \eta,
\] (7)

and opened the way to many other findings about KdV. These include:

- the existence of infinitely many symmetries: \((1)\) belongs to an infinite family (hierarchy) of nonlinear partial differential equations (NLPDE’s), with similar properties (the KdV hierarchy);
- the existence of exact “\(N\)-soliton” solutions to \((1)\), and to the other members of the KdV hierarchy, which describe particle-like collisions between an arbitrary number \(N\) of sech squared pulses of type \((7)\);
- the existence of a Bäcklund transformation (BT) which relates different solutions of a “potential” version of \((1)\);
- the possibility of solving initial value problems for \((1)\), by means of an “inverse scattering transformation” method.

In short, it was recognized that KdV—a characteristic equation governing weakly nonlinear long waves of a quite general type—is the prototype of a “completely integrable” system [36] with infinitely many degrees of freedom.

Among other such systems represented by a NLPDE in \(1+1\) dimensions, with sech squared soliton solutions and with an associated linear eigenvalue problem, there is the “good” Boussinesq or “nonlinear beam” equation [28, 34]:

\[
Bq_\alpha(u) \equiv \pm 3u_{2t} - \alpha u_{2x} + (u_{2x} + 3u^2)_{2x} = 0, \quad \alpha > 0.
\] (8)

Its integrability properties are linked to those of the more basic equation [4]:

\[
Bq(u) \equiv 3u_{2t} + (u_{2x} + 3u^2)_{2x} = 0,
\] (9)

to which \((8)\) is related by the “shift”: \(u \rightarrow u - \frac{\alpha}{6}\).

These properties are well known [3], as are the links between KdV and Bq, as reductions of a more general three dimensional soliton equation—the KP equation [23]—which belongs to a fundamental integrable hierarchy (the KP hierarchy).

Yet, in spite of the extensive studies devoted to soliton systems in general, and to the above two representatives in particular, it seems in retrospect that only limited attention has been paid to the following basic questions: