A Generalized Quasilinearization Method for Nonlinear Second-Order Impulsive Differential Equations Involving the $p$-Laplacian

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Abstract Using variational method and lower and upper solutions, we get a generalized quasilinearization method which construct an iterative scheme converging uniformly to a solution of a nonlinear second-order impulsive differential equations involving the $p$-Laplacian, and converging quadratically when $p = 2$.

Keywords Variational method · Generalized quasilinearization method · Lower and upper solutions · Impulsive differential equation · $p$-Laplacian

Mathematics Subject Classification (2000) 34B15

1 Introduction

Impulsive differential equations, that is, differential equations involving impulse effects, appear as a natural description of observed evolution phenomena of several real world problems in biology, physics, engineering, etc. For instance, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. In [6], Bainov considered a scalar differential equation of Verhulst:

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right), \quad \Delta x(t_k) = x(t_k^+) - x(t_k^-) = \delta_k,$$

where $r > 0$ and $K > 0$ are called the rate coefficient and the capacity of the environment, $x$ is the quantity of biomass of a certain species of micro-organism cultivated in a bioreactor.

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External on the development of the species can cause jumps in the quantity of biomass $x$. For instance, this is possible with a single removal of part of the biomass or with the introduction of a supplementary quantity of biomass into the bioreactor. As a result of an external effect at the moment $t = t_k$, let the quantity of biomass $x(t)$ suffer an increment $\delta_k$, that is $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = \delta_k$, where $x(t_k^-)$ and $x(t_k^+)$ are respectively the quantities of biomass before and after the impulsive effect.

The $p$-Laplacian operator appears in a lot of research areas. For instance, in Fluid Mechanics, the shear stress $\tau$ and the velocity gradient $\nabla_p u$ of certain fluids obey a relation of the form $\tau = a(x)\nabla_p u$, where $\nabla_p u = |\nabla u|^{p-2}\nabla u$. Here $p > 1$ is an arbitrary real number and the case $p = 2$ (respectively $p < 2$, $p > 2$) corresponds to a Newtonian (respectively pseudoplastic, dilatant) fluid. The resulting equations of motion then involve div$(a\nabla_p u)$, which reduces to $a\Delta_p u = a \text{div}\nabla_p u$, provided that $a$ is a constant. The $p$-Laplacian also appears in the study of torsional creep (elastic for $p = 2$, plastic as $p \to \infty$), flow through porous media when $p = \frac{3}{2}$, nonlinear elasticity when $p \geq 2$ and glaciology when $1 \leq p \leq \frac{4}{3}$, see [3].

To prove existence results for nonlinear problems, the generalized quasilinearization method is an important technique. This method coupled with the lower and upper solutions manifests itself as an effective and flexible mechanism. The generalized quasilinearization method was developed by Lakshmikantham [12–14] and widely applied in [1–5, 7–11, 15–19]. Among these references, Amster and De Nápoli [5] considered the following problem

\[
\begin{align*}
\Delta u &= f(x, u) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \eta} &= g(x, u) \quad \text{on } \partial \Omega
\end{align*}
\]

by applying variational method and the generalized quasilinearization method. Recently, the generalized quasilinearization method was successfully applied to the impulsive differential equations. In [4], Ahmad and Nieto considered an impulsive anti-periodic differential equation by using the generalized quasilinearization method. But to our knowledge, there is no paper to consider the $p$-Laplacian impulsive problem via the generalized quasilinearization method.

In this paper, we consider a nonlinear $p$-Laplacian problem

\[
\begin{align*}
(\Phi_p(u'))' - a\Phi_p(u) &= f(t, u), \quad t \in (0, 1), \; t \neq t_k, \\
u'(0) &= g(u(0)), \quad u'(1) = h(u(1)),
\end{align*}
\] (1.1)

with nonlinear impulsive condition

\[
\Delta \Phi_p(u'(t_k)) = I_k(u(t_k)), \quad k \in \{1, 2, \ldots, m\},
\] (1.3)

where $p > 1$, $a > 0$, $\Phi_p(u) = |u|^{p-2}u$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$. Function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and twice continuously differentiable with respect to $u$, $I_k, g, h \in C^2([\mathbb{R}, \mathbb{R}], u'(t_k^-) = \lim_{\tau \to t_k^-} u'(\tau)$ and $u'(t_k^+) = \lim_{\tau \to t_k^+} u'(\tau)$ exist, $\Delta \Phi_p(u'(t_k)) = \Phi_p(u'(t_k^-)) - \Phi_p(u'(t_k^+))$, $k = 1, 2, \ldots, m$.

We say that $u \in W^{1, p}(0, 1)$ is a solution of BVP (1.1)–(1.3) if it satisfies (1.1), (1.2) and for every $k = 0, 1, \ldots, m$, $u_k = u|_{(t_k, t_{k+1})}$ is such that $u_k \in W^{2, p}(t_k, t_{k+1})$. For $k = 1, 2, \ldots, m$, the limits $u'(t_k^-)$, $u'(t_k^+)$ exist, $u'(t_k^-) = u'(t_k^+)$ and (1.3) holds. Here $W^{1, p}(0, 1)$ is the usual Sobolev space endowed with the norm $\|u(t)\| = (\int_0^1 (|u(t)|^p + |u'(t)|^p)dt)^{\frac{1}{p}}$.

By using variational method and generalized quasilinearization method, we construct a nondecreasing iterative scheme converging uniformly to a solution of (1.1)–(1.3). If $p = 2$, we prove the convergence of the iterative scheme is quadratic.