Qualitative Properties for a Fourth Order Rational Difference Equation

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Abstract In this paper we deal with some properties of the solutions of the recursive sequence

\[ x_{n+1} = ax_{n-1} + \frac{bx_{n-1}x_{n-3}}{cx_{n-1} + dx_{n-3}}, \quad n = 0, 1, \ldots, \]

where the initial conditions \( x_{-3}, x_{-2}, x_{-1}, x_0 \) are arbitrary positive real numbers and \( a, b, c, d \) are constants. Also, we give the form of the solution of some special cases of this equation.

Keywords Stability · Boundedness · Solution of difference equations

Mathematics Subject Classification (2000) 39A10

1 Introduction

The nature of many biological systems naturally leads to their study by means of a discrete variable. Particular examples include population dynamics and genetics. Some elementary models of biological phenomena, including a single species population model, harvesting of fish, the production of red blood cells, ventilation volume and blood CO2 levels, a simple epidemics model, and a model of waves of disease that can be analyzed by difference equations are shown in [29]. Recently, there has been interest in so-called dynamical diseases, which correspond to physiological disorders for which a generally stable control system becomes unstable. One of the first papers on this subject was that of Mackey and Glass [28]. In that paper they investigated a simple first order difference-delay equation that models the concentration of blood-level CO2. They also discussed models of a second class of diseases associated with the production of red cells, white cells, and platelets in the bone marrow.
The dynamical characteristics of population system have been modelled, among others by differential equations in the case of species with overlapping generations and by difference equations in the case of species with non-overlapping generations.

In practice, one can formulate a discrete model directly from experiments and observations. Sometimes, for numerical purposes one wants to propose a finite-difference scheme to numerically solved a given differential equation model, especially when the differential equation cannot be solved explicitly. For a given differential equation, a difference equation approximation would be most acceptable if the solution of the difference equation is the same as the differential equation at the discrete points [26]. But unless we can explicitly solve both equations, it is impossible to satisfy this requirements. Most of the time, it is desirable that a differential equation, when derived from a difference equation, preserves the dynamical features of the corresponding continuous-time model such as equilibria, their local and global stability characteristics and bifurcation behaviors. If such discrete models can be derived from continuous-time models and it will preserve the considered realities; such discrete-time models can be called ‘dynamically consistent’ with the continuous-time models.

The study of asymptotic stability and oscillatory properties of solutions of difference equations is extremely useful in the behavior of mathematical models of various biological systems and other applications. This is due to the fact that difference equations are appropriate models for describing situations where the variable is assumed to take only a discrete set of values and they arise frequently in the study of biological models, in the formulation and analysis of discrete time systems, the numerical integration of differential equations by finite-difference schemes, the study of deterministic chaos, etc. For example, [27] the study of oscillation of positive solutions about the positive steady state \( N \) in the delay logistic difference equation

\[
N_{n+1} = N_n \exp \left[ r \left( 1 - \sum_{j=0}^{m} p_j N_{n-j} \right) \right],
\]

where \( r, p_m \in (0, \infty), p_0, p_1, \ldots, p_{m-1} \in [0, \infty) \) and \( m + r \neq 1 \), which describes situations where population growth is not continuous but seasonal with non-overlapping generations, leads to the study of oscillations about zero of a linear difference equation of the form

\[
x_{n+1} - x_n + \sum_{i=0}^{m} p_i x_{n-k_i} = 0, \quad n = 0, 1, \ldots.
\]

Also, difference equations are appropriate models for describing situations where population growth is not continuous but seasonal with overlapping generations. For example, the difference equation,

\[
y_{n+1} = y_n \exp \left[ r \left( 1 - \frac{y_n}{K} \right) \right],
\]

has been used to model various animal populations. This equation is considered by some to be the discrete analogue of the logistic differential equation

\[
y'(t) = ry(t) \left( 1 - \frac{y(t)}{k} \right),
\]

where \( r \) and \( k \) are the growth rate and the carrying capacity of population, respectively.