Spin(9) and almost complex structures on 16-dimensional manifolds

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Abstract For a Spin(9)-structure on a Riemannian manifold $M^{16}$ we write explicitly the matrix $\psi$ of its Kähler 2-forms and the canonical 8-form $\Phi_{\text{Spin}(9)}$. We then prove that $\Phi_{\text{Spin}(9)}$ coincides up to a constant with the fourth coefficient of the characteristic polynomial of $\psi$. This is inspired by lower dimensional situations, related to Hopf fibrations and to Spin(7). As applications, formulas are deduced for Pontrjagin classes and integrals of $\Phi_{\text{Spin}(9)}$ and $\Phi_{\text{Spin}(9)}^2$ in the special case of holonomy Spin(9).

Keywords Spin(9) · Spin(7) · Octonions · Kähler form

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1 Introduction

Although Spin(9) belongs to M. Berger’s list in his holonomy theorem, it has been known for a long time that the only simply connected complete Riemannian manifolds with holonomy Spin(9) are the Cayley projective plane $\mathbb{O}P^2 = \text{Spin}^+(9)$ and its dual, the Cayley hyperbolic plane $\mathbb{O}H^2 = \text{Spin}^-(9)$ (cf. [5,12], as well as [10, Chapter 10]). It is also known that, on the unique irreducible 16-dimensional Spin(9)-module $\Delta_9$, the space $\Lambda^8$ of exterior 8-forms contains a 1-dimensional invariant subspace $\Lambda^8_1$. Thus, any generator of $\Lambda^8_1$ can be viewed as
a canonical 8-form $\Phi_{\text{Spin}(9)}$ on $\mathbb{R}^{16}$, which is Spin(9)-invariant with respect to the standard Spin(9)-structure.

In the same year 1972 when the quoted paper [12] by Brown and Gray appeared, Berger published an article [9] on the Riemannian geometry of rank one symmetric spaces, containing the following very simple definition of a Spin(9)-invariant 8-form $\Phi_{\text{Spin}(9)}$ in $\mathbb{R}^{16}$:

$$
\Phi_{\text{Spin}(9)} \overset{\text{def}}{=} c \int_{\mathbb{P}^1} p_l^* v_l \, dl.
$$

(1.1)

Here $v_l$ is the volume form on the octonionic lines $l = \{ (x, mx) \}$ or $l = \{ (0, y) \}$ in $\mathbb{O}^2 \cong \mathbb{R}^{16}$, $p_l : \mathbb{O}^2 \to l$ is the projection on $l$, the integral is taken over the “octonionic projective line” $\mathbb{O}P^1 = S^8$ of all the $l \subset \mathbb{O}^2$ and $c$ is a normalizing constant. In the same article, Berger writes a similar definition: $\Phi_{\text{Sp}(n) \cdot \text{Spin}(1)} \overset{\text{def}}{=} c \int_{\mathbb{P}^{n-1}} p_l^* v_l \, dl$ for a quaternionic 4-form in $\mathbb{H}^n \cong \mathbb{R}^{4n}$. Note that such definitions of $\Phi_{\text{Spin}(9)}$ and $\Phi_{\text{Sp}(n) \cdot \text{Spin}(1)}$ arose from distinguished 8-planes or 4-planes in the two geometries, appearing thus very much in the spirit of (at the time forthcoming) calibrations. It is also worth reminding that the stabilizers of $\Phi_{\text{Spin}(9)}$ in GL(16, $\mathbb{R}$) and of $\Phi_{\text{Sp}(n) \cdot \text{Spin}(1)}$ in GL(4n, $\mathbb{R}$) are precisely the subgroups Spin(9) and Sp(n) · Sp(1), respectively (cf. [15, pp. 168–170] and [28, p. 126]).

The paper by Brown and Gray contains a different definition of $\Phi_{\text{Spin}(9)}$, as a Haar integral over Spin(8). A natural question is whether an explicit and possibly simple algebraic expression of $\Phi_{\text{Spin}(9)}$ can be written in $\mathbb{R}^{16}$, in parallel with the usual definitions of the G2-invariant 3-form $\Phi_{\text{G}_2}$ on $\mathbb{R}^7$ or the Spin(7)-invariant 4-form $\Phi_{\text{Spin}(7)}$ on $\mathbb{R}^8$ (see for example the books [22] and [23]).

Indeed, some such algebraic expressions have already been written. Namely, Abe and Matsubara computed $\Phi_{\text{Spin}(9)}$ obtaining its 702 terms from the triality principle of Spin(8) (see [1] and [2], and note that some of the terms have to be corrected [3]). More recently, a different approach has been presented by Castrillon Lopez et al. [14], where a detailed exam is given for the invariance of elements of $\Lambda^8(\mathbb{R}^{16})$ under the generators of the group Spin(9).

A major progress in understanding Spin(9)-structures came in the context of weak holonomies by the work of Friedrich: in [17] and [18] it is observed that the number of possible “weakened” holonomies Spin(9) is 16, exactly like in the cases of the groups U(n) and G2, and also that a Spin(9)-structure on $M^{16}$ can be described as a certain vector subbundle $V^9 \subset \text{End}(TM)$. This fact suggests a similarity between Spin(9) and the quaternionic group Sp(n) · Sp(1).

More precisely, a Spin(9)-structure is a rank 9 real vector bundle $V^9 \subset \text{End}(TM) \to M$, locally spanned by self-dual involutions $\mathcal{I}_\alpha$, for $\alpha = 1, \ldots, 9$, such that $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, for $\alpha \neq \beta$ (cf. Definition 1). From these data, the local almost complex structures

$$
J_{\alpha \beta} \overset{\text{def}}{=} \mathcal{I}_\alpha \circ \mathcal{I}_\beta
$$

(1.2)

are defined on $M^{16}$, and the $9 \times 9$ skew-symmetric matrix of their Kähler 2-forms

$$
\psi \overset{\text{def}}{=} (\psi_{\alpha \beta})
$$

(1.3)

is naturally associated with the Spin(9)-structure. The 36 differential forms $\psi_{\alpha \beta}$, for $\alpha < \beta$, are thus a local system of Kähler two-forms of the Spin(9)-manifold $(M^{16}, V^9)$.

The first result of this article is the explicit computation of the 702 terms of $\Phi_{\text{Spin}(9)}$, according to the work by Abe and Matsubara, and on the grounds of Berger’s definition of