Unipotent Algebraic Affine Supergroups and Nilpotent Lie Superalgebras

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(Received: March 2003; accepted: February 2004)
Presented by S. Montgomery

Abstract. We characterize hereditary (as coalgebras) Hopf algebras by the property of ‘equivariant smoothness’, and apply the result to generalize to the super-context, the category equivalence, due to Hochschild, between the unipotent algebraic affine groups and the finite-dimensional nilpotent Lie algebras, in characteristic zero. The global dimension of commutative Hopf algebras, regarded as coalgebras, is also discussed.

Mathematics Subject Classification (2000): 16W30.
Key words: Hopf superalgebra, affine supergroup, Lie superalgebra, hereditary Hopf algebra, global dimension.

Introduction

Throughout this paper we work over a fixed field \( k \) whose characteristic \( \text{ch} \ k \) will be often supposed to be zero.

By definition, an affine group (scheme) is a representable functor \( G \) from the category of commutative algebras to the category of groups; by Yoneda’s lemma, \( G \) is necessarily represented by a uniquely determined, commutative Hopf algebra. Naturally, much effort has been made to simplify and generalize (or quantize) the theory of affine groups by applying results on (not necessarily commutative) Hopf algebras, such as the freeness theorems, Hopf crossed products and duality. For example, applications of Hopf crossed products to classical results on affine groups can be found in [M].

Recall here Hochschild’s theorem [H2, Theorem XVI 4.2], which states that if \( \text{ch} \ k = 0 \), the Lie algebra functor \( G \mapsto \text{Lie}(G) \) gives an equivalence from the category of unipotent algebraic affine groups to the category of finite-dimensional nilpotent Lie algebras; see Serre [Se, Chapitre VII] for previous results on commutative unipotent algebraic groups. The main objective of this paper is to generalize this theorem to corresponding algebraic systems in the symmetric tensor category, \( \delta \text{Vec} \), of supervector spaces. Those spaces are by definition the vector spaces...
graded by the group $\mathbb{Z}_2 = \{0, 1\}$, which indeed form a symmetric tensor category with the obvious tensor product and a specific symmetry; see Section 2 below. Algebraic systems in $\mathcal{S}\text{Vec}$ are called with ‘super’ prefixed, and especially, Lie superalgebras have been intensively studied; see Kac [Kac] and Scheunert [S], for example. The interest in Hopf superalgebras is recently increasing; see Kostant [K], Boseck [B], Andruskiewitch et al. [AEG] and Scheunert et al. [SZ], for example.

If we replace in Hochschild’s theorem above, ‘affine groups’ and ‘Lie algebras’ with ‘affine supergroups’ and ‘Lie superalgebras’, respectively, what is obtained is precisely our main result, Theorem 3.2. But, our proof using induction seems new even in the ungraded context; it is purely Hopf algebraic and rather self-contained. Besides standard techniques such as duality and smash (co)product constructions, an important ingredient is to apply Theorem 1.2; it proves, as a direct consequence of results from [AMS], [M], that a Hopf algebra $H$ is hereditary if and only if any inclusion $H \hookrightarrow C$ of left (or right) $H$-module coalgebras such that $H \supset \text{Corad}(C)$ splits. Here $H$ is said to be hereditary [NTZ], if the global dimension $\text{gl.dim}(H)$ of the coalgebra $H$, or the supremum of the injective dimensions of all $H$-comodules, is at most 1. Theorem 1.2 just referred to gives a true explanation of ‘equivariant smoothness’ for those Hopf algebras which are not necessarily cocommutative. The dual result is specialized to the commutative context by Proposition 1.10; it implies that if a commutative Hopf algebra $H$ is smooth, any epimorphism $A \twoheadrightarrow H$ of commutative right (or left) $H$-comodule algebras with nilpotent kernel splits. This will be applied in Proposition 2.4 to strengthen, though slightly, a fundamental result [B, Theorem 1] on finitely generated commutative Hopf superalgebras in characteristic zero.

Finally in Section 4, we apply Hochschild’s theorem to describe the global dimension of a finitely generated commutative Hopf algebra in characteristic zero; see Proposition 4.2.

1. Characterization of Hereditary Hopf Algebras

Let $C$ be a coalgebra. Let $\mathcal{M}_C$ (resp., $^C\mathcal{M}$) denote the abelian category of right (resp., left) $C$-comodules, which has enough injectives. As is proved by Năstăescu, Torrecillas and Zhang [NTZ], the right and the left global dimensions of $C$

\begin{align*}
\text{rgl.dim}(C) &= \sup\{\text{inj.dim}(V) \mid V \in \mathcal{M}_C\}, \\
\text{lgl.dim}(C) &= \sup\{\text{inj.dim}(V) \mid V \in ^C\mathcal{M}\}
\end{align*}

coincide, where inj.dim means the injective dimension. The coinciding value is denoted by $\text{gl.dim}(C)$, called the global dimension of $C$. $\text{gl.dim}(C) = 0$ if and only if $C$ is cosemisimple.

DEFINITION 1.1 [NTZ]. $C$ is said to be hereditary if $\text{gl.dim}(C) \leq 1$, or equivalently if any quotient comodule of an arbitrary injective in $\mathcal{M}_C$ or $^C\mathcal{M}$ is injective.