Graded Modules Over the $q$-Analog Virasoro-Like Algebra

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Abstract In this paper, we deal with the classification of the irreducible $\mathbb{Z}$-graded and $\mathbb{Z}^2$-graded modules with finite dimensional homogeneous subspaces for the $q$ analog Virasoro-like algebra $L$. We first prove that a $\mathbb{Z}$-graded $L$-module must be a uniformly bounded module or a generalized highest weight module. Then we show that an irreducible generalized highest weight $\mathbb{Z}$-graded module with finite dimensional homogeneous subspaces must be a highest (or lowest) weight module and give a necessary and sufficient condition for such a module with finite dimensional homogeneous subspaces. We use the $\mathbb{Z}$-graded modules to construct a class of $\mathbb{Z}^2$-graded irreducible generalized highest weight modules with finite dimensional homogeneous subspaces. Finally, we classify the $\mathbb{Z}^2$-graded $L$-modules. We first prove that a $\mathbb{Z}^2$-graded module must be either a uniformly bounded module or a generalized highest weight module. Then we prove that an irreducible nontrivial $\mathbb{Z}^2$-graded module with finite dimensional homogeneous subspaces must be isomorphic to a module constructed as above. As a consequence, we also classify the irreducible $\mathbb{Z}$-graded modules and the irreducible $\mathbb{Z}^2$-graded modules with finite dimensional homogeneous subspaces and center acting nontrivial.

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1 Introduction

The classification of graded irreducible modules with finite dimensional homogeneous subspaces over a graded Lie algebra is one of the main subjects in the study of Lie theory. Meanwhile, the irreducible graded modules with nonzero center and finite dimensional homogeneous subspaces for some infinite dimensional Lie algebras have important applications in the study of the vertex operator algebras and theoretical physics. In this paper, we study the graded irreducible modules with finite dimensional homogeneous subspaces for the $q$ analog Virasoro-like algebra $L$, and classify such modules with nonzero center. The $q$ analog Virasoro-like algebra is introduced by Kirkman etc in [11]. It can be realized as the universal central extension of the inner derivation Lie algebra of the quantum torus $\mathbb{C}_q[x_1^\pm 1, x_2^\pm 2]$ (see [3], or [10]), where $q$ is generic. The quantum torus is one of the main objects in noncommutative geometry, and plays an important role in the classification of extended affine Lie algebras [3]. Meanwhile, the $q$ analog Virasoro like algebra can be regarded as a $q$ deformation of the Virasoro-like algebra introduced and studied by Arnold, de Wit, etc when they try to generalize the Virasoro algebra ([1, 6, 11, 13–16] and [17]). There are some papers devoted to the study of structure and representations of the $q$ analog Virasoro algebra $L$. C. Jiang and D. Meng studied its derivation Lie algebra and the automorphism group of its derivation Lie algebra [10]. We studied the structure of its automorphism group [5]. K. Zhao and E. Rao constructed a class of highest weight irreducible $\mathbb{Z}$-graded modules, and gave a sufficient and necessary condition for such a module with finite dimensional homogeneous subspaces [7]. Y. Gao constructed a class of principal vertex representations for the extended affine Lie algebras coordinatized by certain quantum tori by using the representation of $q$ analog Virasoro-like algebra [9]. In the present paper, we deal with the classification of irreducible graded $L$-modules with finite dimensional homogeneous subspaces by using the results on the irreducible modules of Heisenberg algebra obtained by V. Chari (see [4], also [8]), and the results about the $\mathbb{Z}$-graded $L$ modules and the $\mathbb{Z}^2$-graded $L$ modules given in [7] and [12].

The paper is arranged as follows. In Section 2 we recall the concepts of the $q$-analog Virasoro-like algebra and its graded modules. We also collect some results about the irreducible modules of the Heisenberg algebra which is crucial in the study of irreducible representations for the Lie algebra $L$. In Section 3 we first prove that a $\mathbb{Z}$-graded $L$-module must be either uniformly bounded, or a generalized highest weight module. Then we complete the classification of the irreducible generalized highest weight $\mathbb{Z}$-graded modules with finite dimensional homogeneous subspaces. Finally, in Section 4 we first construct a class of irreducible generalized highest weight $\mathbb{Z}^2$-graded modules with finite dimensional homogeneous subspaces by using the $\mathbb{Z}$-graded modules studied given in the previous section. Then we prove that an irreducible generalized highest weight $\mathbb{Z}^2$-graded $L$-modules must be isomorphic to the module constructed in the beginning of the section.