The Dixmier-Moeglin Equivalence for Cocommutative Hopf Algebras of Finite Gelfand-Kirillov Dimension

Jason P. Bell · Wing Hong Leung

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Abstract Let $k$ be an algebraically closed field of characteristic zero and let $H$ be a noetherian cocommutative Hopf algebra over $k$. We show that if $H$ has polynomially bounded growth then $H$ satisfies the Dixmier-Moeglin equivalence. That is, for every prime ideal $P$ in $\text{Spec}(H)$ we have the equivalences

$$P \text{ primitive } \iff P \text{ rational } \iff P \text{ locally closed in } \text{Spec}(H).$$

We observe that examples due to Lorenz show that this does not hold without the hypothesis that $H$ have polynomially bounded growth. We conjecture, more generally, that the Dixmier-Moeglin equivalence holds for all finitely generated complex noetherian Hopf algebras of polynomially bounded growth.

Keywords Primitive ideals · Nullstellensatz · Gelfand-Kirillov dimension · Cocommutative Hopf algebras · Dixmier-Moeglin equivalence

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1 Introduction

The Dixmier-Moeglin equivalence is a fundamental result in the representation theory of enveloping algebras of finite-dimensional Lie algebras. Generally speaking, the problem of finding the irreducible representations of an algebra is a very difficult and sometimes intractable problem. As part of a program to deal with this issue, Dixmier proposed that one should first find the kernels of irreducible representations—these are called primitive ideals of the algebra and they form a distinguished subset of the prime spectrum. This would at least provide a rough classification of the equivalence classes of the irreducible representations. This approach has been very fruitful in the study of enveloping algebras and their representations. The problem of characterizing the primitive ideals of enveloping algebras of finite-dimensional complex Lie algebras was solved completely by Dixmier [3] and Moeglin [15]. In this case, we have the following equivalence.

**Theorem 1.1** Let \( L \) be a finite-dimensional complex Lie algebra and let \( P \) be a prime in \( \text{Spec}(U(L)) \). Then the following are equivalent:

1. \( P \) is primitive;
2. \( \{P\} \) is locally closed in \( \text{Spec}(R) \);
3. \( P \) is rational.

We recall that a set is locally closed in a topological space if it is of the form \( X \setminus Y \), for some closed subsets \( X \) and \( Y \) of the ambient set; in the case that the set we are considering is a singleton \( \{x\} \), we will say that \( x \) is locally closed rather than writing \( \{x\} \) is locally closed. A prime \( P \) of a noetherian algebra \( A \) is rational if the centre of the artinian ring of quotients of \( A/P \) is an algebraic extension of the base field. (Some authors insist that the centre in fact be a finite extension of the base field, although the property is most commonly considered when dealing with algebras over an algebraically closed base field, and there is no distinction in this case.) The centre of the artinian ring of quotients of a prime noetherian algebra \( R \) is called the extended centroid of \( R \), and we denote it by \( C(R) \). Thus we see that the Dixmier-Moeglin equivalence gives both a topological and a purely algebraic characterization of the annihilators of simple modules.

In general, any noetherian algebra with the property that (1)–(3) are equivalent for all prime ideals of the algebra is said to satisfy the Dixmier-Moeglin equivalence. To be precise, we give the following definition.

**Definition 1.2** Let \( R \) be a noetherian algebra. We say that \( R \) satisfies the Dixmier-Moeglin equivalence if for every \( P \in \text{Spec}(R) \), the following properties are equivalent:

1. \( P \) is primitive;
2. \( P \) is rational;
3. \( P \) is locally closed in \( \text{Spec}(R) \).

Since the work of Dixmier and Moeglin, the Dixmier-Moeglin equivalence has been shown to hold for many classes of algebras, including many quantized coordinate rings and quantized enveloping algebras, algebras that satisfy a polynomial identity, and many algebras that come from noncommutative projective geometry [1, 2, 6, 18]. The Dixmier-Moeglin equivalence is known not to hold in general, however. Even in the case of noetherian cocommutative Hopf algebras (of which enveloping algebras of finite-dimensional Lie algebras form an important subclass), it is known that the Dixmier-Moeglin equivalence...