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STABLE VALUED FIELDS

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We are concerned with a class of valued fields, called stable. We propound an extension of a notion in the monograph by S. Bosch, U. G"untzer, and R. Remmert (Non-Archimedean Analysis. A Systematic Approach to Rigid Analytic Geometry, Springer, Berlin (1984)), namely, that of a (ultrametric) norm on groups, rings, algebras, and vector spaces, to the case where the value of the norm is taken from an arbitrary (not necessarily Archimedean) linearly ordered Abelian group (using — as in the general theory of valuations — the version of a logarithmic norm). Our main result extends Proposition 6 in the cited monograph to the general case, thereby making it possible to use the technique of Cartesian spaces to deliver further results on stable valued fields.

In the paper we deal with a class of valued fields, which have different names (for definition of a valued field, see [1]). In [2], such fields are termed defectless; in [3], stable. Here, we opt for the latter term.

For the case of Henselian valued fields, the corresponding property in [4] was called algebraic completeness. In [5], stable fields were defined (for valued fields with Archimedean value groups) in terms of Cartesian spaces with norm, and it was proved that this concept is equivalent to being defectless (see [5, Sec. 3.6, Prop. 6]).

In the present paper we come up with an extension of the notion of a (ultrametric) norm on groups, rings, algebras, and vector spaces, as defined in [5], to the case where the value of the norm is taken from an arbitrary (not necessarily Archimedean) linearly ordered Abelian group (using — as in the general theory of valuations — the version of a logarithmic norm; see [6]). We define the concepts of being orthogonal and of being Cartesian, preserving all the basic properties specified in [5].

Our main result is Theorem 2, which extends [5, Sec. 3.6, Prop. 6] to the general case. This theorem allows us to use the machinery of Cartesian spaces propounded in [5] to obtain further results on stable valued fields.

I. Let \( \langle \Gamma, +, \leq, 0 \rangle \) be a linearly ordered Abelian group and \( \Gamma^\omega = \Gamma \cup \{\omega\} \) be the semigroup obtained by adjoining an element \( \omega \), for which we put \( a + \omega = \omega + a = \omega, a \leq \omega \) for all \( a \in \Gamma \), and \( \omega + \omega = \omega \). If \( \langle G, +, 0 \rangle \) is an Abelian group, then a (ultrametric) (\( \Gamma \)-)norm on \( G \) is any mapping \( v : G \to \Gamma^\omega \) such that:

1. \( v(g) = \omega \Leftrightarrow g = 0, g \in G \);
2. \( v(g_0 - g_1) \geq \min\{v(g_0), v(g_1)\} \).

If \( R = \langle R, +, \cdot, 0, 1 \rangle \) is a ring then a (\( \Gamma \)-)norm on \( R \) is any mapping \( v : R \to \Gamma^\omega \) such that \( v \) is a norm on the group \( \langle R, +, 0 \rangle \), and the following conditions hold:

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(3) \( v(ab) \geq v(a) + v(b) \);
(4) \( v(1) = 0 \) (or, equivalently, \( v(1) \geq 0 \)).

If (3) is satisfied in a strong form, that is, \( v(ab) = v(a) + v(b) \) for all \( a, b \in R \), then \( v \) is called a \((\Gamma-)\)valuation of the ring \( R \).

Let \( R \) be a ring, \( M \) be an \((\text{left})\) \( R \)-module, and \( v_0 : R \to \Gamma^w \) and \( v_1 : M \to \Gamma^w \) be norms. We say that \( M \) is a \textit{normed} \( R \)-module if \( v_0(am) \geq v_0(a) + v_1(m) \) for all \( a \in R \) and \( m \in M \). Norms on a field \( F \) are exemplified by valuations of \( F \) (see [1]).

Let \( F = (F, R) \) be a valued field, \( \bar{F} = (\bar{F}, \bar{R}) \geq F \), and \( \bar{F} \) be an algebraic closure of \( F \). We define an \((\bar{F}-)\)\textit{spectral norm} \( ||_\sigma : \bar{F} \to \Gamma^w_R \) by setting

\[
|\alpha|_\sigma = \min\{v_{\bar{R}}(\varepsilon(\alpha)) \mid \varepsilon \in G(\bar{F}/F)\} \text{ for } \alpha \in \bar{F}.
\]

Now we verify that a spectral norm is a norm. To do this, we argue for the following:

**LEMMA 1.** Let \( \alpha \in \bar{F} \) and \( f_\alpha = x^n + a_1x^{n-1} + \ldots + a_n \in F[x] \) be a minimal polynomial for \( \alpha \) over \( F \). Then

\[
|\alpha|_\sigma = \min \left\{ \frac{1}{k}v_{\bar{R}}(a_k) \mid 1 \leq k \leq n \right\}.
\]

Let \( f_\alpha = \prod_{i<n} (x - \alpha_i) \) be a factorization of the polynomial \( f_\alpha \) over \( \bar{F} \) into linear factors and suppose that \( v_{\bar{R}}(a_0) \leq v_{\bar{R}}(a_1) \leq \ldots \leq v_{\bar{R}}(a_{n-1}) \). Choose \( k \leq n \) so that \( v_{\bar{R}}(a_0) = \ldots = v_{\bar{R}}(a_{k-1}) \) and \( v_{\bar{R}}(a_{k-1}) < v_{\bar{R}}(a_k) \), if \( k < n \). We verify that \( \frac{1}{k}v_{\bar{R}}(a_k) = v_{\bar{R}}(a_0) \leq \frac{1}{k}v_{\bar{R}}(a_i) \) for all \( 0 \leq i < n \).

For any \( 0 < i < n \) and any \( 0 \leq j_0 < j_1 < \ldots < j_{k-1} < n \), we have \( v_{\bar{R}}(a_0) \leq v_{\bar{R}}(a_0 \ldots a_{j_i}) \leq v_{\bar{R}}(a_0 \ldots a_{j_{k-1}}) \). In view of \( a_i = \sum \pm \alpha_{j_0} \ldots \alpha_{j_{k-1}} \), we obtain \( v_{\bar{R}}(a_i) \geq v_{\bar{R}}(a_0) = iv_{\bar{R}}(a_0) \). For the \( k \)-chosen, we also have \( v_{\bar{R}}(a_{j_0} \ldots a_{j_{k-1}}) \geq v_{\bar{R}}(a_0 \ldots a_{k-1}) = kv_{\bar{R}}(a_0) \), for all \( 0 \leq j_0 < j_1 < \ldots < j_{k-1} < n \) such that \( (j_0, \ldots, j_{k-1}) \neq (0, \ldots, k-1) \); hence \( v_{\bar{R}}(a_k) = v_{\bar{R}}(a_0 \ldots a_{k-1}) = kv_{\bar{R}}(a_0) \). \( \Box \)

**COROLLARY 1.** For any \( \alpha \in F^\times \) and any \( \beta \in \bar{F} \), the following equalities hold:

\[
|\alpha|_\sigma = v_{\bar{R}}(\alpha), \quad |\alpha \beta|_\sigma = v_{\bar{R}}(\alpha) + |\beta|_\sigma.
\]

We verify property (2) in the definition of a norm. Let \( \alpha \neq \beta \in \bar{F} \) and let \( \sigma \in G(\bar{F}/F) \) be such that \( |\alpha - \beta|_\sigma = v_{\bar{R}}(\tau(\alpha - \beta)) \). We have

\[
v_{\bar{R}}(\tau(\alpha - \beta)) = v_{\bar{R}}(\tau(\alpha) - \tau(\beta)) \geq \min\{v_{\bar{R}}(\tau(\alpha)), v_{\bar{R}}(\tau(\beta))\} \geq \min\{|\alpha|_\sigma, |\beta|_\sigma\}.
\]

All the other properties are also readily verified as above.

**Remark 1.** Lemma 1 shows that the above definition of a spectral norm coincides with a relevant definition in [5].

**Remark 2.** A spectral norm \( ||_\sigma \) on \( \bar{F} \) coincides with a valuation \( v_{\bar{R}} \) if and only if \( F \) is a Henselian valued field.

This remark allows us to construct norms on fields which would differ from valuations.

Further, we consider the following situation. Let \( \mathbb{R} = (R, v) \) be a \( \Gamma \)-valued ring (i.e., \( v : R \to \Gamma^w \) is a valuation of the ring \( R \)), \( M \) be an \( R \)-module, and \( w : M \to \Gamma^w \) be a \( \Gamma \)-norm on the additive group \( M \) such that \( w(rm) = v(r) + w(m) \) for all \( r \in R \) and \( m \in M \). Then we say that \( \mathbb{M} = (M, w) \) is an \( \mathbb{R} \)-module.

A family \( \alpha_0, \ldots, \alpha_{n-1} \) of elements of the \( \mathbb{R} \)-module \( \mathbb{M} \) is said to be \( \mathbb{R} \)-orthogonal if, for any \( r_0, \ldots, r_{n-1} \in R \), the following equality holds:

\[
w \left( \sum_{i<n} r_i \alpha_i \right) = \min \{w(r_i \alpha_i) = v(r_i) + w(\alpha_i) \}.
\]

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