ISOTOPES OF PRIME \((-1,1)\)-
AND JORDAN ALGEBRAS

S. V. Pchelintsev

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We deal with adjoint commutator and Jordan algebras of isotopes of prime strictly \((-1,1)\)-algebras. It is proved that a system of identities of the form 
\[ [x_1, x_2, x_2, x_3, \ldots, x_n] \]
for \(n = 2, \ldots, 5\) is discernible on isotopes of prime \((-1,1)\)-algebras. Also it is shown that adjoint Jordan algebras for suitable isotopes of prime \((-1,1)\)-algebras may possess distinct sets of identities. In particular, isotopes of a prime Jordan monster have different sets of identities in general.

INTRODUCTION

An idea of obtaining new objects from old ones by using derivative operations has long been known in algebra [1]. In its most general form, the idea was realized by A. I. Mal’tsev [2]. Let \(M_n\) be an associative algebra of matrices of order \(n\) over a field \(\Phi\). Assume that some finite collection \(\Lambda = (a_{ij}, b_{ij}, c_{ij})\) of matrices in \(M_n\) is given. Denote by \(M_n^{(\Lambda)}\) an algebra defined on a space of matrices in \(M_n\) with respect to new multiplication \(x \cdot y = \sum_{i,j} a_{ij} x b_{ij} y c_{ij}\). In [2], it was proved that every \(n\)-dimensional algebra over \(\Phi\) is isomorphic to a subalgebra of \(M_n^{(\Lambda)}\).

The concept of an isotope was introduced by Albert [1]. Let algebras \(A\) and \(A_0\) have a common linear space on which right multiplication operators \(R_x\) and \(R_x^{(0)}\) are defined (for \(A\) and \(A_0\), resp.). We say that \(A_0\) and \(A\) are isotopic if there exist invertible linear operators \(\varphi, \psi,\) and \(\xi\) such that \(R_x^{(0)} = \varphi R_x \psi \xi\). And we call \(A_0\) an isotope of \(A\).

Isotopes are also of interest in the particular case where \(\varphi, \psi,\) and \(\xi\) are operators of multiplication by fixed elements of a given algebra [2]. In the present paper, we look at isotopes of just this kind (\(e\)-isotopes). (See Sec. 1 for exact definitions.)

The notion of isotopy plays an important part in quasigroup theory; specifically, it was applied with advantage in dealing with Moufang loops in [3].


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In the theory of alternative and Jordan algebras, there are a number of results underpinned by the idea of using $c$-isotopes. Thus, for instance, homotopes were applied in the theory of Jordan algebras in constructing a quasiregular radical [4, Chap. 14] and in studying Jordan matrix algebras [5]. I. P. Shestakov employed isotopes to give a negative solution to a radical splitting problem for a finite-dimensional right alternative algebra. In [6], isotopes were made use of in proving that there exist exceptional prime alternative algebras distinguishing among identities of Lie nilpotence $l_n := [x_0, x_1, \ldots, x_n]$ of index $n = 1, 2, 3$.

Isotope identities for alternative and $(-1,1)$-algebras were explored in [7-10]. In [7, 8], it was proved that under certain restrictions, isotopes of an alternative algebra inherit identities of the algebra. In [9, 10], it was shown that a variety generated by a free $(-1,1)$-algebra of rank 2 is closed under taking homotopes, and that an isotope of a free $(-1,1)$-algebra of rank 3 inherits identities of the algebra.

In this paper, we study isotopes of prime strictly $(-1,1)$-algebras—more exactly, adjoint commutator and Jordan algebras for these isotopes. It is proved that the adjoint algebra $(A^{(c)})^{-}$ of the homotope $A^{(c)}$ of a strictly $(-1,1)$-algebra $A$ satisfies the following identities: $[x, y, y, x] = 0$ (Thm. 1) and $l_6 := [x_0, x_1, \ldots, x_6]$ (Thm. 2). In particular, $(A^{(c)})^{-}$ is a binary Lie algebra. It is also shown that a system of identities of the form $[x_1, x_2, x_3, \ldots, x_n]$ ($n = 2, \ldots, 5$) is discernible on isotopes of prime $(-1,1)$-algebras (Thm. 3).

We are unaware whether the estimate specified in Theorem 2 is exact; nor do we know if $(A^{(c)})^{-}$ is a Mal’tsev algebra. On the other hand, an isotope of a prime $(-1,1)$-algebra is generally not a binary $(-1,1)$-algebra (Prop. 1).

In [11], Zel’manov proved a celebrated structure theorem for prime nondegenerate Jordan algebras. In [12], it was shown that there exist prime degenerate Jordan algebras, which arise as adjoints of prime strictly $(-1,1)$-algebras.—We might well prove that all the prime degenerate Jordan algebras specified in [12], when treated over a field of characteristic 0, have equal sets of identities.

For adjoint Jordan algebras, we come up with the following results.

Every homotope $A^{(c)}$ for a strictly $(-1,1)$-algebra $A$ satisfies the following identities: $w_{x,y}^2 = (w_{x,y}, x, y) = 0$, where $w_{x,y} = 2xR_{y,y}^+ \odot x - x^2R_{y,y}^+$ (Thm. 4).

If $A$ is a prime nonassociative $(-1,1)$-algebra, then an identity of the form $(w_{a,b}, x, x) = 0$ holds in the adjoint algebra $A^{+}$ but fails in the isotope $A^{(c)}$, where $c = 1 + c_0$, $c_0$ is an absolute zero divisor of $A$ of order of order at least 3 (Thm. 5).

The main results of the paper—Theorems 3 and 5—are analogs of the theorem on isotopes for prime alternative algebras given in [6]. Note also that isotopes may be used to find new central functions, i.e., those assuming nonzero values in the commutative center of a free strictly $(-1,1)$-algebra. Below we specify two such functions: namely,