GENERALIZED ANALYTIC SETS

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We study completely valued algebraically closed fields in languages with integral analytic functions. It is shown that a nontrivial model theory of integral functions can be developed on a condition that a language is chosen properly.

1. PRELIMINARIES AND FORMULATION OF THE MAIN RESULTS

In the present article we deal with a model theory of analytic functions in the field $\mathbb{C}$ and, which is of special interest, in algebraically closed fields with non-Archimedean valuations. The problem of studying fields in the languages $L_\mathcal{F}$ with names for some collection $\mathcal{F}$ of integral functions including $+$ and $\cdot$, with the natural interpretation of these symbols, is thought of as topical to current research. The field of complex numbers in the language $\mathcal{L}_\mathcal{F} = \{\exp, +, \cdot\} = \exp$ is a canonical example, which is far from being trivial. Clearly, in $\mathbb{C}_\exp$, the ring of integers can be interpreted. Despite the obvious undecidability, the model theory for that structure can well be “analyzed.” In [1], in particular, the following question was posed: Does $\mathbb{C}_\exp$ have automorphisms different from the identity or complex conjugation? Although this question is still unsettled, in a more general setting, the answer for it is negative. A widely known example due to Fatu and Bieberbach is that of two integral functions $f_1$ and $f_2$ of two variables such that the image of $\mathbb{C}^2$ under $(f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ is an open set whose complement also contains an open subset. Actually, $f_1$ and $f_2$ are obtained via the superposition of $+\cdot$, and some integral function $\varphi$ of two variables. For a complex line $L$ on the plane $\mathbb{C}^2$, the intersection $V$ of that line with the range of the map $(f_1, f_2)$ is an open subset of $L$, and its complement in $L$ also contains an open subset of $L$. We can assume that $L$ coincides with $\mathbb{C}$ and that $V$ is bounded. It follows that the structure $\mathbb{C}_\varphi$, whose index corresponds to the language $\{\varphi, +, \cdot\}$, has only continuous automorphisms, that is, the identity and the conjugation.

We can show that Mycielski’s problem has a positive solution if the following question has an affirmative answer: Does $\mathbb{C}_\exp$ have the quasiminimality property? In other words, we ask if every definable (with parameters) subset of the structure is countable or the complement of a countable subset. For the Fatu-Bieberbach example, the answer is obviously negative. Thus, if we attempt the most general approach, the probability of coming across interesting results is unlikely to be high. To develop the model theory of integral functions, we can use algebraic geometry, provided that the latter is considered as a theory of positively definable subsets in some structures. A canonical example is an $n$-dimensional projective space

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P^n(K) over the algebraically closed field K, with predicates corresponding to closed (in the sense of Zariski) subsets of Cartesian powers of the universe. It is well known that every positively definable relation (set) will be Zariski closed in a given structure. In other words, what we have is a "positive quantifier elimination." It is also worth noting that analytic functions — most often integral or of a type close to them — are of considerable importance in algebraic geometry. The task of gaining a clear insight into the role of analytic (transcendency) methods in algebraic geometry is an explanation of why we are interested in the present topic.

In particular, we will be treating structures in which the universe is P^1(K) and the signature is L_F, where F consists of names of integral functions on K. Our interpretation of the latter, as will be shown below, is slightly different from the standard one. Let K be an algebraically closed valued field, complete in the given valuation. For such fields, there does exist a well-developed analytic theory, which covers the case of non-Archimedean valuations. For that case, based on the ideas of Grothendieck and Tate, the so-called rigid analytic geometry was developed; see [2, 3].

Before giving an exact interpretation of the signature L_F, for the sake of convenience, we introduce some changes in the conventional notation. Instead of P^1(K), write \( \hat{K} \), where \( \hat{K} = K \cup \{\infty\} \), and so \( \hat{K}^n \) denotes the nth direct power. The symbol \( F \in \hat{L}_F \), corresponding to the integral function \( f : K^n \to K \) in \( F \), is interpreted as a subset \( F \) of the space \( \hat{K}^{n+1} \), defined by the following conditions:

(i) \( F \) is a closure in the Zariski topology of graph \( (f) \), the graph of the function \( f \), if \( f \) is a polynomial;
(ii) \( F \cap \hat{K}^{n+1} = \text{graph}(f) \) and \( F \setminus \hat{K}^{n+1} = \{ (z_1, \ldots, z_{n+1}) \in \hat{K}^{n+1} : z_1 = \infty \lor \ldots \lor z_{n+1} = \infty \} \) in other cases.

Such subsets \( F \subseteq \hat{K}^{n+1} \) generally are not analytic. For them, however, the following condition is satisfied: \( F \) has a subset \( F' \) analytic in \( \hat{K}^{n+1} \), and \( F \setminus F' \) is analytic in \( \hat{K}^{n+1} \setminus F' \). (In case (i), \( F' = \emptyset \), and in case (ii) \( F' = F \setminus \hat{K}^{n+1} \).) The situation can be generalized by using the following inductive definition.

Definition. A subset \( S \subseteq \hat{K}^{n+1} \) is called generalized analytic (g.a.) in \( \hat{K}^{n+1} \) if \( S \) is analytic in \( \hat{K}^{n+1} \), or there exists \( S' \subseteq S \) such that \( S' \) is generalized analytic in \( \hat{K}^{n+1} \) and \( S \setminus S' \) is analytic in \( \hat{K}^{n+1} \setminus S' \).

The basic results of this article are the following three theorems.

THEOREM A. The class of generalized analytic subsets (predicates) of spaces \( \hat{K}^n \) (for all natural \( n \)) is closed under positive logical operations: direct product, \( \lor, \land, \exists \), and \( \forall \). Moreover, the closure of an arbitrary subclass of generalized analytic subsets, which contains all singletons under a direct product, \( \lor, \land, \) and \( \exists \), is closed also under \( \forall \).

The theorem is proved for both C and the non-Archimedean case.

The next theorem is more interesting for the case of non-Archimedean valuations since, for the field C, the statement follows immediately from the fact that g.a. subsets of \( \hat{C}^n \) are closed in a classical topology of the field C.

THEOREM B. For every tuple \( F \), the structure \( (\hat{K}, \hat{L}_F) \) is atomically compact, that is, for every \( n \), the topology on \( \hat{K}^n \), the basis of closed subsets of which consists of positively definable subsets in the signature \( \hat{L}_F \), is compact.

Note that \( \hat{K}^n \) cannot be compact in the topology induced by a non-Archimedean valuation (since \( K \) is algebraically closed) but is "compact" in the Grothendieck topology (see [2]), which treats only "admissible" open subsets and coverings. Theorem B is of crucial importance if we want to follow the approach outlined in [4], since it provides the existence of a specialization \( \pi \) such that classical analytic functions are at least continuous in the \( \pi \)-topology. We do not succeed, however, in proving that these are analytic in the sense of [4].