A DYNAMICAL DECOMPOSITION THEOREM

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Abstract. By the decomposition theorem \( \dim X \leq n \) if and only if \( X \) admits a decomposition into \( n+1 \) zero-dimensional subspaces \( Z_i \) for \( i = 0, \ldots, n \). If \( f : X \to X \) is a homeomorphism, then under some dimensional restrictions on the set of periodic points, the \( Z_i \) can be chosen to be images of \( Z_0 \) under iterates of \( f \).

1. Introduction

The classical decomposition theorem of dimension theory says that a nonempty metric space has dimension \( n \) if and only if it can be decomposed as a union of \( n+1 \) zero-dimensional subspaces \([1, 2]\). In this paper we study the following problem. For a homeomorphism \( f : X \to X \) on a metric space of dimension \( \dim X = n \), does there exist a zero-dimensional subspace \( A \subset X \) such that \( X = A \cup f(A) \cup \ldots \cup f^{n-1}(A) \)? Obviously, if \( f \) is the identity and \( \dim X > 0 \), then it is impossible to decompose the space in this way. So some condition on the periodic points of \( f \) is required. Theorem 8 gives conditions which are necessary and sufficient.

All spaces are assumed to be metrizable and the dimension function is the covering dimension. A union \( X = A_0 \cup A_1 \cup \ldots \cup A_n \) is called a decomposition of \( X \) and, if the \( A_i \) are pairwise disjoint, the decomposition is called a partition. We shall assume that the space \( X \) is nonempty. We first prove our main theorem and then we give some applications.

2. Decomposing a space into homeomorphic zero-dimensional spaces

**Theorem 1.** Suppose that \( \dim X \leq n \) and suppose that \( f : X \to X \) is a homeomorphism such that \( f \) has no points of period \( \leq n \). Then there exists a dense \( G_\delta \)-subset \( Z \subset X \) with \( \dim Z \leq 0 \) and

\[
X = Z \cup f(Z) \cup \ldots \cup f^n(Z).
\]

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The proof requires some preparation. For each \( i \in \mathbb{N} \) let \( \{ U_\alpha \mid \alpha \in A_i \} \) be a locally finite cover of \( X \) that refines the collection of all open \( 1 \) balls. Let \( \{ F_\alpha \mid \alpha \in A_i \} \) be a closed shrinking. We may assume that \( F_\alpha \neq \emptyset \) for all \( \alpha \in A_i \). We may also assume that each index-set \( A_i \) is well-ordered and that the \( A_i \) are pairwise disjoint. The union of all the index-sets \( A = \bigcup A_i \) can be well-ordered by defining that \( a_i < a_j \) for elements \( a_i \in A_i \) and \( a_j \in A_j \) whenever \( i < j \). In this way we get an open collection \( \mathcal{U} = \{ U_\alpha \mid \alpha \in A \} \) and a closed shrinking \( \mathcal{F} = \{ F_\alpha \mid \alpha \in A \} \) with the following property: if \( \mathcal{V} = \{ V_\alpha \mid \alpha \in A \} \) is a family such that \( F_\alpha \subset V_\alpha \subset U_\alpha \) for each \( \alpha \in A \), then \( \mathcal{V} \) is a \( \sigma \)-locally finite basis. Because of this property, we call \(( \mathcal{F}, \mathcal{U} )\) a framework for open bases of \( X \). Note that for each \( \beta \in A \), the collection \( \{ U_\alpha \mid \alpha < \beta \} \) is locally finite.

**Proposition 2.** A metric space contains a dense zero-dimensional \( F_\sigma \)-subset.

**Proof.** Let \(( \mathcal{F}, \mathcal{U} )\) be a framework for open bases and let \( \mathcal{F} = \bigcup \{ F_\alpha \mid \alpha \in A \} \) be as in the definition of a framework. Choose \( x_\alpha \in F_\alpha \) for each \( \alpha \in A_i \). It is clear that the collection of points \( C_i = \{ x_\alpha \mid \alpha \in A_i \} \) is a zero-dimensional closed subset. The union of the \( C_i \) is a dense zero-dimensional \( F_\sigma \). \( \Box \)

The idea of the proof of our main theorem follows. Suppose that \(( \mathcal{F}, \mathcal{U} )\) is a framework of \( X \). In a very special way we shall select sets \( S_\alpha \) which separate \( F_\alpha \) and \( X \setminus U_\alpha \). The union \( S = \bigcup \{ S_\alpha \mid \alpha \in A \} \) is a \( F_\sigma \)-subset of \( X \) and its complement \( Z = X \setminus S \) is zero-dimensional. The special way in which we select the separating sets ensures that \( S \cap f(S) \cap \ldots \cap f^n(S) = \emptyset \) or equivalently \( S_\alpha \cap f(S_{\alpha_1}) \cap \ldots \cap f^n(S_{\alpha_n}) = \emptyset \) for each possible choice of indices \( \alpha_i \). So \( Z \) has the property that \( X = Z \cup f(Z) \cup \ldots \cup f^n(Z) \).

The special way in which we select the separators \( S_\alpha \) involves a process of peeling off zero-dimensional \( F_\sigma \)-subsets. The following proposition will be employed in the proof.

**Proposition 3.** Suppose that \( X \) is an \( n \)-dimensional space. There exists a zero-dimensional \( F_\sigma \)-subset \( F \) such that \( X \setminus F \) is at most \( (n-1) \)-dimensional.

**Proof.** By induction. The result is obviously true if \( X \) is zero-dimensional. Suppose that \( \dim X = n \). Let \(( \mathcal{F}, \mathcal{U} )\) be a framework. For each \( \alpha \in A \) choose a closed \( S_\alpha \) separating \( F_\alpha \) and \( X \setminus U_\alpha \) such that \( \dim S_\alpha \leq n-1 \). The union \( S = \bigcup \{ S_\alpha \mid \alpha \in A \} \) is closed and at most \( (n-1) \)-dimensional, so \( S = \bigcup S_i \) is an \( (n-1) \)-dimensional \( F_\sigma \). Its complement \( G \) is a zero-dimensional \( G_\sigma \), so we can partition the space as \( X = G \cup S \), where \( G \) is a zero-dimensional and \( S \) is an \( F_\sigma \) of dimension \( \leq n-1 \). By induction, we can find a zero-dimensional \( F_\sigma \)-subset \( F \subset S \) such that \( X \setminus F \) is at most \( (n-2) \)-dimensional. By the addition theorem, the complement \( X \setminus F \) is at most \( (n-1) \)-dimensional. \( \Box \)