BETWEEN COMPACTNESS AND QUASICOMPACTNESS
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Abstract. Three weak variants of compactness which lie strictly between compactness and quasicompactness, are introduced. Their basic properties are studied. The interplay with mapping and their direct and inverse preservation under mappings are investigated. In the process three decompositions of compactness are observed.

1. Introduction

Compactness is one of the most important topological properties and plays a prominent role in topology, analysis and many other branches of mathematics. However, in many topological/topologico-analytic situations the full power of compactness is not required. So to meet the demand of a particular situation, an optimal solution is provided by a suitable weak variant of compactness. The known weak variants of compactness with which we shall be dealing in this paper include near compactness [24], almost compactness [2], \(\theta\)-compactness [11], and quasicompactness [5]. Nearly compact spaces were introduced and studied by Singal and Mathur [24]. Hausdorff almost compact spaces are called \(H\)-closed spaces and were introduced by Alexandroff and Urysohn [1] in 1924 and since then have been studied by host of authors (see [4], [8], [15], [16], [19], [20], [21], [22], [30], [31]). The class of quasicompact spaces was initiated by Frolík [5] and has been investigated by Aristotle [3], Stephenson [28], [29] and others. Functionally Hausdorff quasicompact spaces are precisely the spaces in which Stone Weierstrass theorem holds (see [28], [29]). The class of \(\theta\)-compact spaces is studied in [11].

Several such weak variants of compactness occur in the mathematical literature, which partially capture certain features of compactness and have been

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extensively studied by several authors. In this paper we introduce three generalizations of compactness called \(d\)-compactness, \(d^*\)-compactness and \(D_\delta\)-compactness which lie strictly between compactness and quasicompactness.

Section 2 is devoted to preliminaries and basic definitions. In Section 3 basic properties of these weak variants of compactness are studied. Factorizations of compactness are obtained in Section 4. Section 5 is devoted to study the behaviour of \(d\)-compact spaces, \(d^*\)-compact spaces and \(D_\delta\)-compact spaces under mappings, and sufficient conditions for their direct/inverse preservation under mappings are obtained.

2. Preliminaries and basic definitions

2.1. Definition [6]. A collection \(\beta\) of subsets of a space \(X\) is called as open complementary system if \(\beta\) consists of open sets such that for every \(B \in \beta\), there exist \(B_1, B_2, \ldots \in \beta\) with \(B = \bigcup\{X/B_i : i \in \mathbb{N}\}\).

2.2. Definition [6]. A subset \(U\) of a space \(X\) is called a strongly open \(F_\sigma\)-set if there exists a countable open complementary system \(\beta(U)\) with \(U \in \beta(U)\). The complement of a strongly open \(F_\sigma\)-set is called strongly closed \(G_\delta\)-set.

2.3. Definition [18]. A subset \(H\) of a space \(X\) is called a regular \(G_\delta\)-set if \(H\) is an intersection of a sequence of closed sets whose interiors contain \(H\), i.e., if \(H = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} F_0^n\), where each \(F_n\) is a closed subset of \(X\). The complement of a regular \(G_\delta\)-set is called a regular \(F_\sigma\)-set.

2.4. Definition. A space \(X\) is called \(D\)-regular [6] \((D\)-completely regular [13]) if it has a base of open \(F_\sigma\)-sets (strongly open \(F_\sigma\)-sets, regular \(F_\sigma\)-sets).

2.5. Definition [30]. Let \(X\) be a topological space and let \(A \subset X\). A point \(x \in X\) is called a \(\theta\)-limit point of \(A \subset X\) if every closed neighborhood of \(X\) intersects \(A\). Let \(cl_\theta A\) denote the set of all \(\theta\)-limit points of \(A\). The set \(A\) is called \(\theta\)-closed if \(A = cl_\theta A\). The complement of a \(\theta\)-closed set is referred to as a \(\theta\)-open set.

2.6. Definitions. A space \(X\) is said to be

(1) almost compact [2] if every open cover of \(X\) has a finite subcollection the closures of whose members cover \(X\);

(2) quasicompact [5] if every covering of \(X\) by co-zero sets admits a finite subcovering;

(3) nearly compact [24] if every open covering of \(X\) admits a finite subcollection the interiors of the closures of whose members cover \(X\);

(4) \(\theta\)-compact [11] if every cover of \(X\) by \(\theta\)-open sets has a finite subcover.