Abstract. The end conditions for cubic spline interpolation with equidistant knots will be defined so as to make the (slightly modified) B-spline coefficients minimal. This produces good approximation results as compared e.g. with the not-a-knot spline.

1. Introduction

For a natural $n$ let $\Omega_n = \{a + ih, i = 0, \ldots, n\}$ be an equidistant (uniform) partition of the real interval $[a, b]$ with $h = (b - a)/n$. Let $S_3(\Omega_n)$ be the linear space of cubic splines with regard to this partition. Any such spline $s$ can be written uniquely as

$$s = \sum_{i=-3}^{n-1} c_i B_{3,i},$$

where $B_{3,i}$ are the cubic B-splines for the extended knot sequence $\Omega_\infty = \{x_i = a + ih, i \in \mathbb{Z}\}$. For convenience, we give the derivatives of the B-spline $B_{3,i}$ supported in $[x_i, x_{i+4}]$ at the relevant knots in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$x_i$</th>
<th>$x_{i+1}$</th>
<th>$x_{i+2}$</th>
<th>$x_{i+3}$</th>
<th>$x_{i+4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{3,i}(x)$</td>
<td>0</td>
<td>1/6</td>
<td>2/3</td>
<td>1/6</td>
<td>0</td>
</tr>
<tr>
<td>$B'_{3,i}(x)$</td>
<td>0</td>
<td>1/2h</td>
<td>0</td>
<td>-1/2h</td>
<td>0</td>
</tr>
<tr>
<td>$B''_{3,i}(x)$</td>
<td>0</td>
<td>1/h^2</td>
<td>-2/h^2</td>
<td>1/h^2</td>
<td>0</td>
</tr>
</tbody>
</table>
Given a real function $f$ defined in $[a, b]$, the interpolatory conditions $s(x_i) = f(x_i)$, $0 \leq i \leq n$ assume

\[ Mc = \hat{f}, \]

where $M = \frac{1}{6}$ tridiag $(1, 4, 1)$ is an $(n + 1) \times (n + 3)$ matrix, $\hat{f} \equiv f|_{\Omega_n}$ is the column of the function values $(f(x_i))_{i=0}^{n}$, and $c$ is the column of the unknown coefficients $(c_i)_{i=-3}^{n-1}$. We use the notations of [5], Ch. 6.

Our aim is to fix the two end conditions such that the resulting spline

- minimizes the quadratic sum $\|c\|^2 = \sum_{i=-3}^{n-1} c_i^2$ of the coefficients, and
- reproduces the set of cubic polynomials.

Unfortunately, these requirements are conflicting. Hence we will introduce (in the form of a diagonal matrix) further parameters to ‘scale down’ the B-splines, especially the near-end ones.

The method derived has the optimal order of convergence. To prove this, we make use of the properties of the not-a-knot spline [3], cf. [4]: “(ii) It may be possible to carry out the argument by perturbation, ... showing that the change in the side conditions ... is gentle enough (at least for large $n$) to change $\|P''\|$ by a bounded amount...”

The new, (quasi)minimal spline will not bear comparison, of course, with splines using derivative information at the ends; however, it proves to be superior to the not-a-knot spline, as numerical tests suggest.

2. Determining the end conditions

Let the additional unknown rows be $r_a$ and $r_b$, where the subscripts indicate that they are related to $a$ and $b$. Then we get the enlarged system

\[ \begin{pmatrix} r_a \\ M \\ r_b \end{pmatrix} c = \begin{pmatrix} 0 \\ \hat{f} \\ 0 \end{pmatrix} \]

of linear equations with a square matrix.

Our first statement concerns the problem of minimality.

**Lemma 1.** The solution $c$ of (2) is the minimal solution of (1) if and only if $r_a$ and $r_b$ are linearly independent and are orthogonal to the rows of $M$, i.e.

\[ r_a M^T = 0, \quad r_b M^T = 0. \]