PRIMITIVE INVOLUTION RINGS

K. I. BEIDAR* (Tainan), L. MÁRKI (Budapest), R. MLITZ (Wien) and R. WIEGANDT (Budapest)†

Abstract. A $*$-primitive involution ring $R$ is either a left and right primitive ring or a certain subdirect sum of a left primitive and a right primitive ring with involution exchanging the components. An example is given of a left and right primitive ring which admits no row and column finite matrix representation. We characterize $*$-primitive involution rings in terms of maximal $*$-biideals. A $*$-prime involution ring has a minimal left ideal if and only if it has a minimal $*$-bideal, and these involution rings are always $*$-primitive.

1. Introduction

The most natural examples of rings can be endowed with an involution. Let us mention here two kinds of examples of division rings with involution. On a free associative algebra with more than one generator one can define an involution (we shall present this explicitly in Example 3.2): this algebra embeds into a division ring, and there the involution of the free algebra extends to the division subring generated by the free algebra. Next, it is well known that the enveloping algebra of a finite-dimensional Lie algebra over a field admits an involution, and this extends to the classical division ring of quotients of the enveloping algebra. The classical reference for involution rings is, of course, Herstein’s book [10]; some important results can be found in [4]; for a recent treatise on central simple algebras with involution see [14].

From a categorical point of view, the consistent way of looking at the class of involution rings is to consider between them only involution preserving ring homomorphisms. In describing the structure of involution rings in terms within this category, it is a typical feature that an involution ring with a given property (e.g. $*$-simple, $*$-prime, $*$-subdirectly irreducible) is either...

---

*Professor Konstantin I. Beidar (National Cheng Kung University, Tainan) passed away as this paper was close to be finished.
†Research supported by the Austro-Hungarian Bilateral Intergovernmental Cooperation project # A-24/2000 and by the Hungarian OTKA Grants # T034530 and # T043034.
Key words and phrases: involution, primitive and prime ring, biideal.
2000 Mathematics Subject Classification: primary 16W10, secondary 16D60, 16N60.
a ring with that property (e.g. simple, prime, subdirectly irreducible) or a (sub)direct sum of a ring with that property and its opposite with the exchange involution. For instance, a $*$-simple involution ring is either a simple ring or a direct sum of two simple rings – this seems to have been stated first by Jacobson [13] in the special case of finite-dimensional central simple (associative) algebras. Another example: a $*$-subdirectly irreducible involution ring is either a subdirectly irreducible ring or a subdirect sum of two subdirectly irreducible rings subject to further constraints (cf. [9]). Involutive versions of the Wedderburn–Artin Structure Theorems [1], the Litoff–Ánh Theorem [2], Goldie’s Theorems [8], and the description of simple involution rings with a minimal $*$-biideal by Rees matrix rings [3] are also of that kind.

The involutive version of primitivity, called $*$-primitivity, was introduced by Rowen [17], and he noted: “There does not seem to be a good $*$-analog for the density theorem in general, although there is an excellent version for $*$-primitive rings having minimal left ideals”. Nevertheless, one can describe the structure of $*$-primitive rings by distinguishing between cases. In fact, as an extension of an observation of Rowen (see Proposition 2.1 below) we show here that a $*$-primitive involution ring is either a left and right primitive ring or a subdirect sum of two anti-isomorphic (left and right, respectively) primitive rings endowed with an involution which exchanges the components (Theorem 2.4). We also show that not every left and right primitive ring can be represented by row and column finite matrices (Theorem 3.1).

Moreover, the notion of left ideals is alien to the category of involution rings, since a left ideal closed under involution is an ideal. The notion of $*$-biideals is more suitable in the category of involution rings and involution preserving homomorphisms. For instance, chain conditions imposed on $*$-biideals have proved to be efficient in describing the structure of certain involution rings (see [1], [5], [8], [15]). We show that an involution ring $R$ is $*$-primitive exactly when $R^2 \neq 0$ and $R$ possesses a maximal $*$-biideal which does not contain nonzero $*$-ideals of $R$ (Theorem 4.3). We prove that for $*$-prime rings the existence of minimal left ideals is equivalent to the existence of minimal $*$-biideals (Theorem 5.1), and so $*$-prime rings with minimal one-sided ideals have got a description in terms intrinsic to the category of involution rings. We also prove that a $*$-prime ring with minimal $*$-biideals is always $*$-primitive (Theorem 5.3).

Thanks are due to Pham Ngoc Ánh for several useful remarks, among others, for calling our attention to the paper [11], and to the referee for spotting some inaccuracies in the first version.