ON THE PRIME GRAPH OF \( PSL(2, p) \) WHERE \( p > 3 \) IS A PRIME NUMBER

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Abstract. Let \( G \) be a finite group. We define the prime graph \( \Gamma(G) \) as follows. The vertices of \( \Gamma(G) \) are the primes dividing the order of \( G \) and two distinct vertices \( p, q \) are joined by an edge if there is an element in \( G \) of order \( pq \). Recently M. Hagie [5] determined finite groups \( G \) satisfying \( \Gamma(G) = \Gamma(S) \), where \( S \) is a sporadic simple group. Let \( p > 3 \) be a prime number. In this paper we determine finite groups \( G \) such that \( \Gamma(G) = \Gamma(PSL(2, p)) \). As a consequence of our results we prove that if \( p > 11 \) is a prime number and \( p \neq 1 \mod(12) \), then \( PSL(2, p) \) is uniquely determined by its prime graph and so these groups are characterizable by their prime graph.

1. Introduction

If \( n \) is an integer, then we denote by \( \pi(n) \) the set of all prime divisors of \( n \). If \( G \) is a finite group, then the set \( \pi(|G|) \) is denoted by \( \pi(G) \). Also the set of orders of elements of \( G \) is denoted by \( \pi_e(G) \). Obviously \( \pi_e(G) \) is partially ordered by divisibility. Therefore it is uniquely determined by \( \mu(G) \), the subset of its maximal elements. We construct the prime graph of \( G \) as

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follows: The prime graph \( \Gamma(G) \) of a group \( G \) is the graph whose vertex set is \( \pi(G) \), and two distinct primes \( p \) and \( q \) are joined by an edge (we write \( p \sim q \)) if and only if \( G \) contains an element of order \( pq \). Let \( t(G) \) be the number of connected components of \( \Gamma(G) \) and let \( \pi_1(G), \pi_2(G), \ldots, \pi_{t(G)}(G) \) be the connected components of \( \Gamma(G) \). Sometimes we use the notation \( \pi_i \) instead of \( \pi_i(G) \). If \( 2 \in \pi_i(G) \), then we always suppose \( 2 \in \pi_1 \) and \( \pi_i \), where \( i \geq 2 \) are called the odd component(s) of \( \Gamma(G) \). The connected components of non-abelian simple groups with at least two prime graph components are listed in [10, Tables 1, 2].

The concept of prime graph arose during the investigation of certain cohomological questions associated with integral representations of finite groups. It has been proved that for every finite group \( G \) we have \( t(G) \leq 6 \) [9, 15, 26] and the diameter of \( \Gamma(G) \) is at most 5 [16]. Also Hage [5] determined finite groups \( G \) satisfying \( \Gamma(G) = \Gamma(S) \), where \( S \) is a sporadic simple group (see also [12, 13, 14]). In this paper, we determine finite groups \( G \) such that their prime graph is \( \Gamma(\text{PSL}(2, p)) \), where \( p > 3 \) is a prime number. As a consequence of this result, we prove that if \( p > 11 \) is a prime number and \( p \neq 1 \) (mod 12), then \( \text{PSL}(2, p) \) is uniquely determined by its prime graph and so these groups are characterizable by their prime graphs.

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [1], for example. We use the results of J. S. Williams [26], N. Iyori and H. Yamaki [9] and A. S. Kondrat’ev [15] about the prime graph of simple groups and the results of M. S. Lucido [17] about the prime graph of almost simple groups. We denote by \( (a, b) \) the greatest common divisor of positive integers \( a \) and \( b \). Let \( m \) be a positive integer and \( p \) be a prime number. Then \( |m|_p \) denotes the \( p \)-part of \( m \). In other words, \( |m|_p = p^k \) if \( p^k \parallel m \) (i.e., \( p^k \mid m \) but \( p^{k+1} \nmid m \)).

2. Preliminary results

Remark 2.1. First we give a brief description of the prime graph of \( \text{PSL}(2, p) \). By the result of Dickson [8, p. 213], it follows that

\[
\mu(\text{PSL}(2, p)) = \left\{ p, \frac{p-1}{2}, \frac{p+1}{2} \right\}.
\]

Therefore by assumption, the prime graph of \( \text{PSL}(2, p) \) has three connected components, since \( (p-1)/2, (p+1)/2 \) are odd. Also \( |\mu(\text{PSL}(2, p))| = 3 \), and so every component of \( \Gamma(\text{PSL}(2, p)) \) is a clique (a complete subgraph). We