CONTINUOUS OPERATORS ON ASYMMETRIC NORMED SPACES

C. ALEGRE

Departamento Matemática Aplicada, IUMPA, Universidad Politénica de Valencia,
Camino de Vera s/n, 46022 Valencia, Spain

(Received February 15, 2008; accepted June 17, 2008)

Abstract. If \((X,p)\) and \((Y,q)\) are two asymmetric normed spaces, the set \(LC(X,Y)\) of all continuous linear mappings from \((X,p)\) to \((Y,q)\) is not necessarily a linear space, it is a cone. If \(X\) and \(Y\) are two Banach lattices and \(p\) and \(q\) are, respectively, their associated asymmetric norms \((p(x) = \|x^+\|, q(y) = \|y^+\|)\), we prove that the positive operators from \(X\) to \(Y\) are elements of the cone \(LC(X,Y)\).

We also study the dual space of an asymmetric normed space and finally we give open mapping and closed graph type theorems in the framework of asymmetric normed spaces. The classical results for normed spaces follow as particular cases.

1. Introduction

Let \(X\) be a real linear space. A function \(p : X \to \mathbb{R}^+\) is an asymmetric norm on \(X\) ([13], [15]) if for all \(x, y \in X\) and \(r \in \mathbb{R}^+\),

(i) \(p(x) = p(-x) = 0\) if and only if \(x = 0\),

(ii) \(p(rx) = rp(x)\).

*The author acknowledges the support of the Ministerio de Educación y Ciencia of Spain and FEDER, under grant MTM2006-14925-C02-01 and Generalitat Valenciana under grant GV/2007/198.

Key words and phrases: asymmetric norm, asymmetric normed linear space, cone, quasi-metric, semicontinuous linear map.

2000 Mathematics Subject Classification: 54A05, 54E35, 46A03.
(iii) $p(x + y) \leq p(x) + p(y)$.

The pair $(X, p)$ is called \textit{asymmetric normed linear space}. Asymmetric norms are also called quasi-norms in [9], [3], [22] etc. and nonsymmetric norms in [5].

If $p$ is an asymmetric norm on $X$, then the function $p^{-1}$ defined on $X$ by $p^{-1}(x) = p(-x)$ is also an asymmetric norm on $X$, called the \textit{conjugate} of $p$. The function $p^*$ defined on $X$ by $p^*(x) = \max\{p(x), p^{-1}(x)\}$ is a norm on $X$.

A subset $M$ of a linear space is called \textit{cone} or \textit{semilinear space} if for every $x, y \in M$ and $a \in R^+$, $x + y \in M$ and $ax \in M$. If $p$ is a function on $M$ satisfying the conditions of the definition of an asymmetric norm, $p$ will be also called an asymmetric norm on $M$.

A quasi-metric on a nonempty set $A$ is a function $d : A \times A \to \mathbb{R}^+$ such that

1. $d(a, b) = d(b, a) = 0$ if and only if $a = b$;
2. for every $a, b, c \in A$, $d(a, b) \leq d(a, c) + d(c, b)$.

Each quasi-metric $d$ on $A$ generates a $T_0$ topology $T(d)$ on $A$, which has as basic open sets the $d$-balls

$$B_d(a, r) = \{ b \in A : d(a, b) < r \}, \quad a \in A, \ r > 0.$$

The reader may consult [10] and [18] for more information about quasi-metric spaces.

An asymmetric norm $p$ on a linear space $X$ induces the quasi-metric $d_p$ by means of the formula

$$d_p(x, y) := p(y - x), \quad x, y \in X.$$

The $d_p$-ball $B_{d_p}(x, r)$, will be simply denoted by $B_p(x, r)$ and the topology $T(d_p)$ will be denoted by $\tau_p$. Thus, the sets

$$B_p(0, \varepsilon) = \{ x \in X : p(x) < \varepsilon \}, \quad \varepsilon > 0,$$

define a fundamental system of neighborhoods of zero for the topology $\tau_p$, and for all $y \in X$, the sets $B_p(y, \varepsilon) = y + B_p(0, \varepsilon)$ define a fundamental system of neighborhoods of $y$. The terms $p$-neighborhood, $p$-open, $p$-closed, etc., will refer to the corresponding topological concepts with respect to that topology.

In [9] it is exhibited a natural class of examples of asymmetric normed spaces, the normed linear lattices. In fact, it is proved that whenever $(X, \|\cdot\|)$ is a normed lattice, then $p(x) = \|x^+\|$ with $x^+ = \sup\{x, 0\}$ is an asymmetric norm on $X$. Moreover $p$ determines the topology and order of $X$ in the sense of [10]. This class of asymmetric normed spaces is the most interesting one from the point of view of applications.