A SIDON BASIS

J.-M. DESHOUILLERS\textsuperscript{1} and A. PLAGNE\textsuperscript{2}

\textsuperscript{1} Institut mathématique de Bordeaux, UMR 5251, Université de Bordeaux et CNRS, 33405 Talence Cedex, France
e-mail: jean-marc.deshouillers@math.u-bordeaux1.fr

\textsuperscript{2} Centre de Mathématiques Laurent Schwartz, UMR 7640 du CNRS, École polytechnique, 91128 Palaiseau Cedex, France
e-mail: plagne@math.polytechnique.fr

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Abstract. We construct a Sidon set which is an asymptotic additive basis of order at most 7.

1. Introduction

A Sidon set \cite{14} is a set of non-negative integers \( \mathcal{A} \) which has the property that if \( a_1 + a_2 = a_3 + a_4 \) with \( a_1, a_2, a_3, a_4 \in \mathcal{A} \) then \( \{a_1, a_2\} = \{a_3, a_4\} \). In other words, two sums of the form \( a + a' \) with \( a, a' \in \mathcal{A} \) never coincide nontrivially.

While it is known \cite{6} that a finite Sidon set \( \mathcal{A} \) included in \( \{1, \ldots, n\} \) has a cardinality \( |\mathcal{A}| \leq \sqrt{n} \) and that this result cannot be improved \cite{15, 1, 2}, what is proved in the case of infinite Sidon sets is much less precise.

On the one hand, Erdős (a result published in Stöhr’s article \cite{16}) has shown that for any Sidon set \( \mathcal{A} \), one has

\begin{equation}
\liminf_{n \to +\infty} \frac{|\mathcal{A} \cap \{1, \ldots, n\}|}{\sqrt{n / \log n}} \ll 1.
\end{equation}

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In the other direction, Mian and Chowla [10] proved by a greedy recursive construction the existence of an infinite Sidon set satisfying

\[(2) \quad |\mathcal{A} \cap \{1, \ldots, n\}| \geq n^{1/3} \text{ for all } n.\]

Up to now, the best result is that of Ruzsa [11] who proved that the exponent 1/3 in (2) may be replaced by any real number less than \(\sqrt{2} - 1\).

The present paper originates in a question of Erdős, Sárközy and Sós (Problem 14 in [5]; see also Problem 32 in [13] where 2 must be replaced by 3): does there exist an infinite Sidon set which is an asymptotic (additive) basis of order 3? (Order 2 cannot hold, by Erdős’ result (1) just above or simply by considering the finite case.) More modestly, we may ask: does there exist an integer \(h \geq 3\) and an infinite Sidon set which is an asymptotic basis of order \(h\) that is, such that \(hA\) (the set of sums of \(h\) not necessarily distinct elements of \(A\)) contains all large enough integers? Such a set, if it exists, is called a Sidon basis.

There is indeed no reason why a set \(A\) with the Sidon property could not be an asymptotic basis: on the contrary, a Sidon set \(A\) which has the property that \(|2A|\) is as large as possible with respect to \(|A|\), is likely to behave in a nice way, additively speaking.

It is a conjecture of Erdős that the set of fifth powers is a Sidon set (see Problem F30 in Guy’s book [8]). If true, this would answer our question with a value of \(h\) less than or equal to \(G(5)\), which is known to be at most 17 (see [17]), and expected to be equal to 6.

Another possible path to a satisfactory answer would be to use another of Ruzsa’s constructions. Indeed, in [12], it is shown that under certain conditions, sets of the form \(\{n^\alpha + cn^\beta\}, n \in \mathbb{N}\) (with \(\alpha > 4, \beta\) and \(c\) some real numbers subject to certain constraints) are Sidon. Therefore we are reduced to show that such a set is an asymptotic basis; this is doable, by using techniques similar to those of [4] but the order obtained is then quite large.

In this paper, we use an elementary approach, in the vein of Mian-Chowla’s, in order to exhibit a Sidon set which is at the same time an asymptotic basis. Moreover we give an upper bound of reasonable size for the minimal order of such a Sidon basis.

**Theorem 1.** There exists a Sidon set which is an asymptotic additive basis of order at most 7.

How far can this result be improved upon? Although Grekos et al. [7] showed that a Sidon basis of asymptotic order 3 cannot represent all the integers larger than 39, we still believe that the following conjecture is true.

**Conjecture 2.** There exists a Sidon set which is an asymptotic additive basis of order 3.